Linear Algebra
<u>Vectors and Matrices</u>

Fundamental Operations with Vectors

Vector: a directed line segments that has both magnitude and direction

- *notation*: $\vec{v} = \langle v_1, v_2, v_3, \ldots, v_n \rangle = [v_1, v_2, v_3, \ldots, v_n]$= ***v*** where v_1, v_2, …, v_n are the components of the vector.
- if A(x_1, x_2, …x_n) is the initial point of the vector and B(y_1, y_2, …, y_n) is the terminal point then the components can be found using $\vec{AB} = [y_1 - x_{1,} y_2 - x_{2,} \ldots, y_n - x_{n,}]$
- graphically, a vector is drawn as an arrow such that the length of the arro is the magnitude of the vector and the arrow points in the direction of the vector

A vector represents a single entity of magnitude and direction, only one vector is unique.

The **zero vector** ($\vec{0}$): the unique vector that has no magnitude and no direction, every component is zero: $\vec{0} = [0, 0, 0, \ldots\ldots, 0]$.

The **magnitude of a vector** : given some vector ***v*** with intial point A(x_1, x_2, …x_n) and terminal point B(y_1, y_2, …, y_n) , the magnitude of ***v*** denoted $\|\vec{v}\|$ is the distance from A to B:

$$\left\|\vec{AB}\right\| = \|\vec{v}\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} .$$

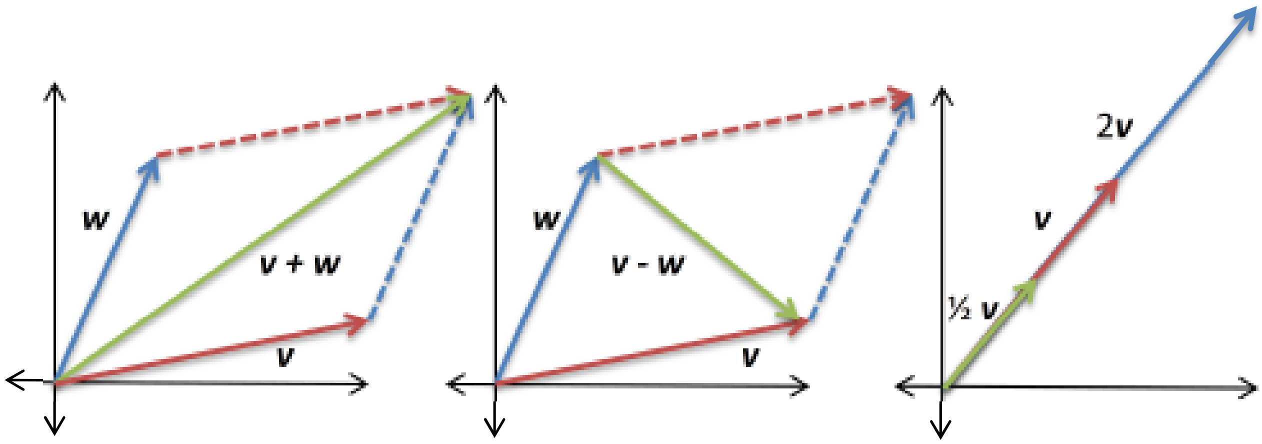
The **unit vector**: any vector ***u*** is a unit vector if its magnitude is 1.
The **standard unit vectors**: any vector e_n is a standard unit vector for $\Re^n$ is every component of e_n is a zero except the nth component, which is a 1.

2-D	3-D	n-D
i = [1,0]	i = [1,0,0]	e_1 = [1, 0,0,0…..,0]
j = [0,1]	j = [0,1,0]	e_2 = [0,1,0,0…..,0]
	k = [0,0,1]	e_3 = [0,0,1,0…..,0]
		e_n = [0, 0,0,0…..,1]

Operation of Vectors:

Let ***v***, ***w***, and ***z*** be vectors in $\Re^n$ and λ be a scalar (constant). Then we define:

- the sum of ***v*** and ***w*** as the vector $\vec{v} + \vec{w} = [v_1, v_2, \ldots, v_n] + [w_1, w_2, \ldots, w_n] = [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]$
- the scalar product of λ and ***v*** is the vector $\lambda\vec{v} = \lambda[v_1, v_2, \ldots, v_3] = [\lambda v_1, \lambda v_2, \ldots, \lambda v_3]$
- the difference of ***v*** and ***w*** is the vector $\vec{v} - \vec{w} = [v_1, v_2, \ldots, v_n] - [w_1, w_2, \ldots, w_n] = [v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n]$
- the opposite of ***v*** is the vector $-\vec{v} = -[v_1, v_2, \ldots, v_n] = [-v_1, -v_2, \ldots, -v_n]$



- Note ***v*** and λ***v*** are always parallel (but may be in opposite directions if λ is negative).

From this we see a method to determine if two vectors are parallel: two vectors ***v*** and ***w*** are **parallel** if they are **scalar multiples** of one another.

<u>The Standard Unit Vectors:</u>
Every vector can be written as a linear combination of the standard unit vectors.

Let ***v*** be a vector in $\Re^n$ such that. Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ be the standard unit vectors. Then we see:

$$\vec{v} = [v_1, v_2, \ldots, v_n] = [v_1, 0, 0, \ldots, 0] + [0, v_2, 0, \ldots, 0] + [0, 0, \ldots, v_n] =$$
$$= v_1[1, 0, 0, \ldots., 0] + v_2[0, 1, 0, \ldots, 0] + \ldots + v_n[0, 0, \ldots, 1] = v_1\vec{e}_1 + v_2\vec{e}_2 + \ldots + v_n\vec{e}_n$$

Suppose we want to find a unit vector ***u*** in the same direction as a given vector ***v***. How do we do it?

This can be done by using the following formula $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$. This works by the following

Since $\frac{\vec{v}}{\|\vec{v}\|} = \vec{v}\left(\frac{1}{\|\vec{v}\|}\right)$, $\|\vec{u}\| = \left\|\frac{\vec{v}}{\|\vec{v}\|}\right\| = \left\|\vec{v}\left(\frac{1}{\|\vec{v}\|}\right)\right\| = \|\vec{v}\|\left(\frac{1}{\|\vec{v}\|}\right) = 1$ so since the magnitude of ***u*** is 1 and $\frac{\vec{v}}{\|\vec{v}\|}$ is in the direction of ***v*** then ***u*** is in the direction of ***v***.

Ex: Given $\vec{v} = [1, -2, 3]$ $\vec{w} = [-4, 1, 1]$ $\vec{z} = [2, -3, -4]$

1. Find $\|\vec{w}\|$ and $\|\vec{z}\|$
2. $4\vec{v}$
3. $3\vec{w} - 2\vec{v} + 5\vec{z}$
4. The unit vector is the same direction as $\vec{w}$ and another in the same direction as $\vec{v}$.

<u>Properties of Vector Operations:</u>
Let $\vec{v}=[v_1,v_2,...,v_n]$, $\vec{w}=[w_1,w_2,...,w_n]$, and $\vec{z}=[z_1,z_2,...,z_n]$ be vectors in $\Re^n$. Also let λ and k be scalars and $\vec{0}$ be the zero vector in $\Re^n$. Then

- $\vec{v}+\vec{w}=\vec{w}+\vec{v}$ *commutative property for addition*
- $(\vec{v}+\vec{w})+\vec{z}=\vec{v}+(\vec{w}+\vec{z})$ *associative property for addition*
- $\vec{v}+\vec{0}=\vec{0}+\vec{v}=\vec{v}$ *additive identity property, which is zero*
- $\vec{v}+(-\vec{v})=(-\vec{v})+\vec{v}=\vec{0}$ *additive inverse property which is –v*
- $\lambda(\vec{v}+\vec{w})=\lambda\vec{v}+\lambda\vec{w}$ and $(\lambda+k)\vec{v}=\lambda\vec{v}+k\vec{v}$ *distributive laws for scalars*
- $(\lambda k)\vec{v}=\lambda(k\vec{v})=k(\lambda\vec{v})$ *associative property for scalar multiplication*
- $1(\vec{v})=\vec{v}(1)=\vec{v}$ *multiplicative identity property for scalar multiplication, which is 1*

<u>Theorem:</u> Let $\vec{v}\in\Re^n$ and $\lambda\in\Re$. I f λ***v*** = 0, then either λ = 0 or ***v*** = 0

<u>Linear Combination of Vectors:</u> Let $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, …..$\boldsymbol{v}_k$ be vectors in R^n then the vector ***v*** is a <u>linear combination</u> of vectros $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, …, $\boldsymbol{v}_k$ if and only if there exists scalars λ_1, λ_2, …, λ_3 such that $\boldsymbol{v} = \lambda_1\boldsymbol{v}_1, \lambda_2\boldsymbol{v}_2, \ldots, \lambda_k\boldsymbol{v}_k$.

<u>The Dot Product</u>: Let ***v*** and ***w*** be vectors in R^n. Then the dot product of ***v*** and ***w*** is the scalar denoted, $\vec{v}\cdot\vec{w}$ given by: $\vec{v}\cdot\vec{w}=[v_1,v_2,...,v_n]\cdot[w_1,w_2,...,w_n]=v_1w_1+v_2w_2+...+v_nw_n$

- the dot product produces a scalar, not a vector
- the dot product is also known as the inner product
- there are many useful properties that come about from the dot product

<u>Properties of the Dot Product</u>: Let ***v***, ***w***, and ***z*** be vectors in R^n and $\lambda\in\Re$ be a scalar. Then the following are true:

- $\vec{v}\cdot\vec{w}=\vec{w}\cdot\vec{v}$ (commutative law for dot product)
- $\vec{v}\cdot\vec{v}=\|\vec{v}\|^2\geq 0$
- $\vec{v}\cdot\vec{v}=0$ iff $\vec{v}=\vec{0}$
- $\lambda(\vec{v}\cdot\vec{w})=(\lambda\vec{v})\cdot\vec{w}=\vec{v}\cdot(\lambda\vec{w})$
- $\vec{v}\cdot(\vec{w}+\vec{z})=(\vec{v}\cdot\vec{w})+(\vec{v}\cdot\vec{z})$
- $(\vec{v}+\vec{w})\cdot\vec{z}=(\vec{v}\cdot\vec{z})+(\vec{w}\cdot\vec{z})$

Ex: Let $\vec{v}=[3,-1]$, $\vec{w}=[5,2]$ $\vec{z}=[2,-1,3,5]$ $\vec{b}=[-3,-2,1,4]$, find the following if possible.

a. $\vec{v}\cdot\vec{w}$ **b.** $\vec{w}\cdot\vec{v}$ **c.** $\vec{z}\cdot\vec{b}$ **d.** $\vec{v}\cdot\vec{b}$ **e.** $\vec{z}\cdot\vec{z}$ **f.** $\|\vec{z}\|$

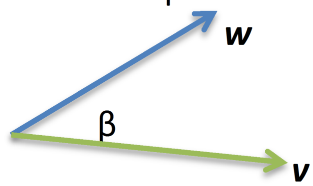
<u>Inequalities Involving the Dot Product</u>

<u>Cauchy-Schwartz Inequality</u>: Let ***v*** and ***w*** be vectors in R^n. Then $|\vec{v}\cdot\vec{w}|\leq\|\vec{v}\|*\|\vec{w}\|$

<u>The Triangle Inequality</u>: Let ***v*** and ***w*** be vector in R^n. Then $\|\vec{v}+\vec{w}\|<\|\vec{v}\|+\|\vec{w}\|$

(both proofs can be found in the textbook)

<u>The Angle between Two Vectors</u>: Let ***v*** and ***w*** be vectors in R^n such that that have a common initial point. Let β be the angle between the vectors,



then $\cos\beta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\| * \|\vec{w}\|}$ or $\beta = \arccos\left(\dfrac{\vec{v}\cdot\vec{w}}{\|v\| * \|w\|}\right)$, also $\vec{v}\cdot\vec{w} = \|v\| * \|w\| \cos\beta$

- two vectors ***v*** and ***w*** are orthogonal (perpendicular) if and only if $\vec{v}\cdot\vec{w} = 0 \left(\beta = \pi/2\right)$
- $0 \le \beta \le \pi$
- when β is 0 or π (cosβ = ±1), ***v*** and ***w*** are parallel

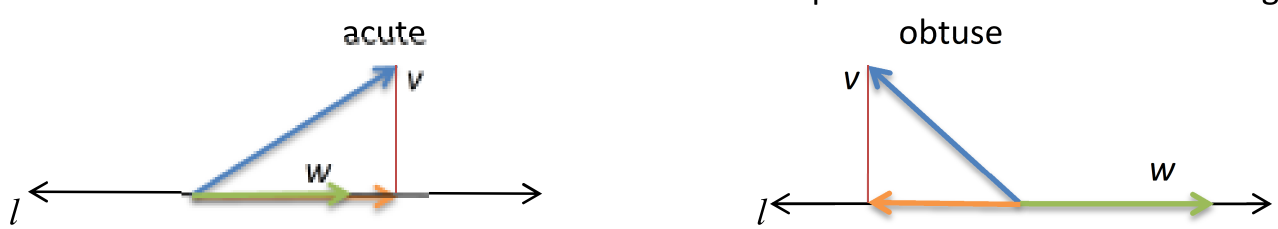
Theorem: Let ***v*** and ***w*** be vectors in R^n. Then ***v*** and ***w*** are parallel if and only if

$$\vec{v}\cdot\vec{w} = \pm\|\vec{v}\| * \|\vec{w}\|$$

Ex: Find the angle between $\langle 3,-1,2\rangle$ and $\langle 1,-1,-2\rangle$

<u>Projection Vectors:</u>

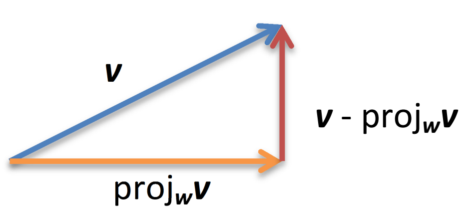
Let ***v*** and ***w*** be vectors in R^n that have the same initial point. Let l be a line containing ***w***.



The projection of ***v*** onto ***w*** (proj$_{\boldsymbol{w}}$***v***) is the vector with same intial point as ***v*** and ***w*** and found by dropping perpendicular from the terminal point of ***v*** onto the line l containing ***w*** to create to terminal point of the projection. Therefore the projection of ***v*** onto ***w*** is a scalar multiple of ***w*** and can be found by:

$$proj_{\vec{w}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\right)\vec{w}$$

We can also decompose the vector v using the projection vector and the perpendicular



Notice that proj$_{\boldsymbol{w}}$***v*** is parallel to ***w*** and ***v*** - proj$_{\boldsymbol{w}}$***v*** is orthogonal to ***w***

Ex: Given **v** = [3, - 2, 1] and **w** = [4, 1, -5] find proj$_{\mathbf{v}}$**w** and proj $_{\mathbf{w}}$**v,** then find the vector orthogonal to ***w***

$\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}$ is the scalar known as the **scalar component of *v*** in the same direction of ***w*** .

Fundamental Operation of Matrices

Matrices: a matrix is a rectangular array of real numbers arranged in m rows and n columns. We say the size or dimension of the matrix is mxn. In general if matrix A is mxn, then:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & & . \\ . & . & . & \cdots & . \\ . & . & . & & . \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- We use capital letters to denote matrices. We can think of an mxn matrix A as a collection of m row vectors of n column vectors.
- We typically use a_{ij} to represent the real number in the i^{th} row and j^{th} column.
- M_{mn} represents to set of all mxn matrices
- The diagonal entries are all a_{11}, a_{22}, a_{33}…

Special Matrices:

A **square matrix** B is a matrix with the same number of rows and columns. Therefore the size of B is either mxm or nxn.

A **diagonal matrix** D is a square matrix such that all the entries not on the main diagonal are zeros. We use $\mathbf{D_n}$ to represent the set of all nxn diagonal matrices.

An **identity matrix** I is a diagonal matrix such that all the main diagonal entries are ones. We use $\mathbf{I_n}$ to represent the nxn identity matrix.

An **upper triangular matrix** is a square matrix such that all entries below the main diagonal entries are zeros. $\mathbf{U_n}$ represents all nxn triangular matrices.

A **lower triangular matrix** is a square matrix such that all entries above the main diagonal are zeros. $\mathbf{L_n}$ represents all nxn lower triangular matrices.

A **zero matrix** is any matrix such that all entire are zeros, denoted 0_{mn}.

Addition of Matrices

Let A and B be mxn matrices The sum of A and B is the mxn matrix whose ij^{th} entry is equal to $a_{ij}+b_{ij}$.

Ex: $\begin{bmatrix} 3 & -1 & 4 & 6 \\ 2 & 5 & -3 & 2 \end{bmatrix}+\begin{bmatrix} 6 & 1 & 7 & 9 \\ -8 & -3 & 6 & 4 \end{bmatrix}=$

Let A be a m×n matrix and let λ be a real number then the m×n matrix λA, the **scalar multiplication of A by λ,** is the matrix whose ijth entry is λa_{ij}.

Ex: $C = \begin{bmatrix} 3 & 2 \\ -1 & 7 \\ 6 & -9 \end{bmatrix}$ find 5C

Properties of Addition and Scalar Multiplication of Matrices

Let A, B, and C be m×n matrices and λ and k be scalars. Then we have:

- A +B = B + A *commutative property*
- (A + B) + C = A + (B + C) *associative property*
- A+ 0_{mn} = 0_{mn} + A = A *existence of an additive identity*
- A + (-1)A = -A + A = 0_{mn} *existence of an additive inverse*
- k(A +B) = kA + kB *distributive laws for scalar multiplication and addition*
- (k + λ)A = kA + λA *distributive laws for scalar multiplication and addition*
- (kλ)A = (kA)λ = (λA)k associativity of scalar multiplication
- 1A = A existence of identity element for scalar multiplication

The Transpose of a Matirx and its Properties:

Let A be any m×n matrix then **the transpose matrix** of A is the m×n matrix A^T such that the ijth entry of A^T is the jith entry from A. *(the rows of A are the columns of A^T and vice versa)*

Ex: Find A^T if A = $\begin{bmatrix} 3 & -1 & 7 & -2 & 4 \\ 2 & 5 & 6 & 9 & -3 \end{bmatrix}$

Properties of the Transpose

Let A and B be mxn matrices and k be a scalar then:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(kA)^T = kA^T$

Symmetric and Skew-Symmetric Matrices

A matrix A is a **symmetric matrix** iff $A = A^T$

A matrix B is a **skew-symmetric matrix** iff $B = -B^T$

only square matrices can be symmetric or skew symmetric

Ex: $A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 4 & 6 \\ -1 & 6 & 7 \end{bmatrix}$ is symmetric

Thereom: Every square matrix A can be decomposed uniquely as a sum of a symmetric matrix S and a skew symmetric V. It can be shown that S= ½(A +A^T) and V = ½ (A – A^T).

Matrix Multiplication

If A is a m×n matrix and B is a n×p matrix, then their matrix product AB is the m×p matrix whose ijth entry is the dot product of the ith row of A and the jth column of B.

- The dimension of AB is the (number of rows of A) x (number of columns of B)
- AB exists only is the number columns of A equals the number of rows of B.

AB may exist but BA may not

$$AB = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ . & . & . & . \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1n}b_{n1} \\ a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2n}b_{n2} \\ \dots\dots \\ a_{m1}b_{1p} + a_{m2}b_{2p} + \dots + a_{mn}b_{np} \end{bmatrix}$$

Ex: Let $A = \begin{bmatrix} 3 & 5 \\ -1 & 6 \\ 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & 5 & 6 \\ 3 & -7 & -1 & 0 \end{bmatrix}$ find AB if possible

Ex: For $D = \begin{bmatrix} -2 & 1 \\ 0 & 5 \\ 4 & -3 \end{bmatrix}$ $E = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ $F = \begin{bmatrix} -6 & 3 & 1 \end{bmatrix}$ $G = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$ $H = \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix}$

Find DE, ED, FG, GF, and HD if possible.

- In the special case where AB = BA we say A and B commute
- If A is m×n and I represents and identity matrix with the appropriate number of rows and columns such that AI_n exists or I_mA exists then then $AI_n = A = I_mA$. I is called the multiplicative identity.

Properties of Matrix Products

Supoose that A,B, and C are matirces for which the following sums and product are defined. Let λ be a scalar then:

- (AB)C = A(BC) *associative property*
- A(B+C) = AB +BC *distributive property*
- (A+B)C = AC +BC *distributive property*
- λ(AB) = (λA)B = A(λB) *scalar multiple*

Let A be a nxn matrix then the non-negative powers of A are given by

$$A^0 = I_nA \text{ and for } k\geq 2\ A^k = A^{(k-1)}A$$

If A is a square matrix andn s and t are non-negative integers, then

$$A^{s+t} = A^sA^t \text{ and } (A^s)^t = A^{st} = (A^t)^s$$

If A is mxn and B is nxp then the transpose of the matrix product is $(AB)^T = B^TA^T$

Systems of Linear Equations

Solving Linear Systems using Gaussian Elimination

A **system of m linear equations** in n variables is a collection of m equations each containing a linear combination of the same variables summing to a scalar:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$$

A **particular solution** (s_1, s_2, …,s_n) to a system of m equations is n variables is an ordered n-tuple of values satisfying every equation when substituted for x_1, x_2, …x_n.

A **complete solution** to a system of m equations in n variables is the set of all ordered n-tuples that satisfy every equation.

The coefficients of all the m variables in all the n variables in the m equations can be collected in a matrix called a **coefficient matrix**.

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$ we can also let $$X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}$$ and $$B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_m \end{bmatrix}$$

The above notation can be written as AX=B

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ b_m \end{bmatrix}$$

The augmented matrix is the matrix $\left[A|B\right]$

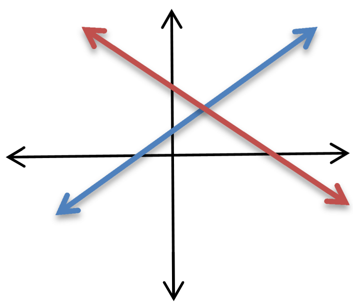
$$\left[A|B\right] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ . & . & . & . & . \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

The Number of Solutions to a System

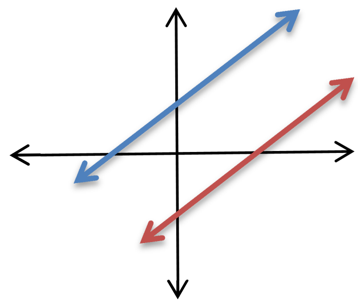
Any given set of equations in s system of linear equations will have one of the following possibilities for the total number of solutions:

- one solution
- no solution
- infinitely many solutions *(if a systems has more than one solution it is guaranteed to have infinitely many solutions)*

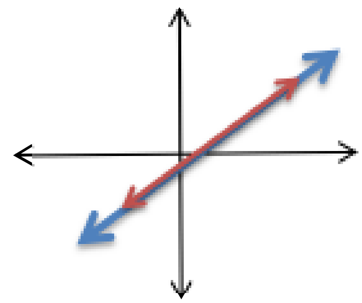
Any system that has a solution is called **consistent** and if there is no solution its **inconsistent**.
Consider a system of two linear equations:



one solution



no solutions



infinitely many solutions

Gaussian Elimination

Gaussian Elimination is a procedure that we use to solve a system. The procedure involves multiplying and entire equation by non-zero constants and adding the equations to another to eliminate a variable *(elimination by addition).* We will use Gaussian elimination on the augmented matrix $\left[A|B\right]$ to solve the systems.

There are three operations we can perform on any given row of a matrix so that the system retains to same solution:

1. multiply a row by a non-zero scalar
2. adding a scalar multiple of one row to another row *(while subtracting is a form of addition and can also be done between rows is very helpful to right operations only using addition)*
3. interchange the positions of any two rows

It is helpful to notate what you are doing to a given row using the previous operations. If we were planning on performing an operation of row 2 by multiplying row 1 by 3 and adding it to row 2 we would write that as: $\mathbf{R_2 = 3r_2 + r_2}$.
If we want to interchange row 1 and row 2 we would write: $\mathbf{R_1 \leftrightarrow R_2}$.

We begin by selecting a specific row of the matrix called the "home row" and a specific entry in that row called the "pivot". We will convert the pivot to a 1 and use the home row and the pivot to zero out the entries above and below the pivot. This process then repeats until a solution is obtained.

Ex: Solve the system

$$\begin{cases} -5x - 2y + 2z = 14 \\ 3x + y - z = -8 \\ 2x + 2y - z = -3 \end{cases}$$

The methods we are using are called **row reduction**. WE are turning the augmented matrix into an upper/lower triangular matrix and then **back substitute**. If we turn the augmented matrix into a diagonal matrix we do not need to back substitute.

Sometimes we have a row or column of all zeros. In these cases no pivot can be chosen and we move on to the next column. If a row consists of all zeros we will move that row to the bottom of the matrix.

$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 6 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & ? \end{array}\right]$ the third row having all zeros to the right of the line amounts to

$$0x+0y+0z=?$$

If ? = 0 there there are two possibilities:

1. infinitely many solutions
2. one solution, the trivial solutions (0,0,….0)

If ? ≠ 0 then there is no solution to the system.
If a column to the right of the augmentation bar has all zeros then there are infinitely many solutions.

Ex: Solve the system

$$\begin{cases} 3x+y+7z+2w=13 \\ 2x-4y+14z-w=-10 \\ 5x+11y-7z+8w=59 \\ 2x+5y-4z-3w=39 \end{cases}$$

Gauss-Jordan Row Reductions and Reduced Row Reduction Form

A matrix in row-echelon form has the following properties:

- Any rows consisting entirely of zeros occur at the bottom of the matrix.
- The entry in row 1, column 1 is a 1 and zeros appear **below** it.
- The first non-zero entry of each row after the first row is a 1 with zeros **below** it.

The disadvantage to this method is we have to back substitute to solve the system

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- Any rows consisting entirely of zeros occur at the bottom of the matrix.
- The entry in row 1, column 1 is a 1 and zeros appear **below** it.
- The first non-zero entry of each row after the first row is a 1 with zeros **below and above** it.

This method does not need back substitution.

Solving a system using row echelon form is called Gaussian elimination and solving a system using reduced row echelon form is called Gauss-Jordan elimination.

Ex: Is the following in reduced row echelon form? Why or why not?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Number of solutions to a System of Linear Equations

Let AX = B be a system of linear equations. And let C be the reduced row echelon form augmented matrix.

- If a column to the right of the augmentation bar has all zeros then there are infinitely many solutions.
- If a row has all zeros to the right of the augmentation bar and a zero to the right there are infinitely many solutions
- If a row has all zeros to the right of the augmentation bar and a non-zero entry to the right there is no solution.
- If none of the previous apply there is a unique solution.

A system of linear equations having the form AX = 0 is called a **homogeneous system.**

A homogenous system will have the following properties:

- If the reduced row echelon form augmented matrix for a homogeneous system in n variables has fewer then n nonzero pivot entries, then the system has only the trivial solution (0,0,…,0).
- Every homogeneous system has at least the trivial solution. Therefore if the system also has a nontrivial solution then it has infinitely many solutions.
- In a homogeneous system if there are more variables then equations then there are infinitely many solutions.

Ex: Solve

$$\begin{cases} 2x_1 + 4x_2 - x_3 + 5x_4 + 2x_5 = 0 \\ 3x_1 + 3x_2 - x_3 + 3x_4 = 0 \\ -5x_1 - 6x_2 + 2x_3 + 6x_4 - x_5 = 0 \end{cases}$$

Equivalent Systems, Rank, and Row Space

Equivalent Systems and Row Equivalence of Matrices:

Two systems of m linear equations in n variables are **equivalent** iff they have the same solution set.

Two matrices are **row equivalent** iff one matrix is obtained from the other by a finite number of row operations.

Each row operation we perform is reversible. To go backwards is called a **reverse** or **inverse row operation**.

If a matrix D is a row equivalent to a matrix C, then the matrix C is a row equivalent to the matrix D.

Let AX = B be a system of linear equations. If $\left[C|D\right]$ is row equivalent to $\left[A|B\right]$ then the system CX = D is equivalent to the system AX = B

Rank of a Matrix:

Every matrix is row equivalent to a unique matrix is reduced row echelon form. SO no matter how we reduce a matrix we will always get the same matrix in reduced row echelon form.

The **rank of a matrix A** is the number of nonzero rows in the reduced row echelon from of A, denoted rank(A).

Ex: Determine the rank of the following:

$$A=\begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 9 \\ 7 & -14 & 21 \end{bmatrix} \qquad B=\begin{bmatrix} 1 & -3 & 5 & 7 & -8 \\ 2 & 5 & -3 & 6 & 1 \\ -8 & -20 & 12 & -24 & 4 \end{bmatrix}$$

Homogeneous Systems and Rank:

Let AX = 0 be a homogeneous system in n variables:

- If rank(A) < n the system has a nontrivial solution ($\therefore$ infinitely many solutions)
- If rank(A) = n the system has only the trivial solution

Corollary: Let AX = 0 be a homogeneous system on m linear equations in n variables. If m < n then the system has a nontrivial solution.

Linear Combinations of Vectors:

Recall: a linear combination of vectors is a sum of scalar multiples of the vectors

Ex: Given $\vec{x}=[3,-1,4,5], \vec{y}=[2,7,-1,6], \textit{ and } \vec{z}=[1,0,0,-3]$ find $3\vec{x}-2\vec{y}+5\vec{z}$

Question: Given a vector, can we determine if the vector is a linear combination of some other vectors?

Answer: To determine whether $\vec{v}$ is a linear combination of $\vec{x}, \vec{y}, \textit{ and } \vec{z}$, we must find C_1, C_2, C_3 such that $\vec{v}=C_1\vec{x}+C_2\vec{y}+C_3\vec{z}$. We can think of this as a system of linear equations, $\begin{cases} x_1C_1+y_1C_2+z_1C_3=v_1 \\ x_2C_1+y_2C_2+z_2C_3=v_2 \\ x_3C_1+y_3C_2+z_3C_3=v_3 \end{cases}$ and use an augmented matrix.

If there is a solution then we can write ***v*** as a linear combination of ***x***, ***y***, and ***z***. IF no solution then we can't.

Ex: Determine whether a $\vec{v}=[2,2,3]$ is a linear combination of

$$x=[6,-2,3],\ \vec{y}=[0,-5,-1],\ \vec{z}=[-2,1,2]$$

Row Space of a Matrix:

Let A be an mxn matrix. The subset of $\Re^n$ consisting of all vectors that are linear combinations of the rows of A is called the **row space of A**.

- Recall that if A is a mxn matrix, then each of the rows of A is a vector with n entries (or in other words a vector in $\Re^n$).
- We understand that the row space is a set of vectors that are all possible linear combinations of the vectors making up the rows of the matrix.
- The zero vector, $\vec{0}$ is guaranteed to be in the row space of every matrix.
- Each row of a matrix is also guaranteed to be in the row space of the matrix.

<u>Lemma</u>: Suppose $\vec{v}$ is a linear combination of the vectors $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_4$ and suppose that each of the vectors $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_4$ is itself a linear combination of the vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_j$. Then $\vec{v}$ is a linear combination of $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_j$.

Suppose that A and B are row equivalent matrices, then the row space of A is equal to the row space of B.

Ex: Determine whether $\vec{v} = \left[4, 0, -3\right]$ is in the row space of $A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & 5 \\ -4 & -3 & 3 \end{bmatrix}$

If ***v*** is in the row space of A then there exists real numbers C_1, C_2, and C_3 such that

$$C_1\left[3,1,1\right] + C_2\left[2,-1,5\right] + C_3\left[-4,-3,3\right] = \vec{v}$$

To check whether a vector ***v*** is in the row space of matrix A we set up the augmented matrix: $\left[A^T \middle| \vec{v}\right]$ the row reduce to see if there is exactly one solution.

<u>Inverse Matrices</u>

Let A be a square nxn matrix, if the inverse of A (denoted A^{-1}) exists then $AA^{-1}=A^{-1}A=I_n$

not all square matrices have inverses (most do!)

A nxn matrix A is **<u>singular</u>** iff the inverse of A <u>does not exist</u>. If the inverse does exists we say the matrix is **<u>non-singular</u>**.

Suppose the nxn matrices B and C are inverses of the nxn matrix A, then B = C. If an inverse exists then that matrix is unique.

Let A be a nxn non-singular matrix. Then we define negative powers of A using the inverse A^{-1} of A. In other words for k ≥ 2 we have $A^{-k} = (A^{-1})^k$.

If A is a nxn non-sigular matrix and s and t are any integers, then we have $A^sA^t=A^{s+t}$ and $(A^s)^t = (A^t)^s = A^{st}$

Let A and B be nxn non-singular matrices, then:

- A^{-1} is non-singular and $(A^{-1})^{-1} = A$
- A^k is non-singular and $(A^k)^{-1} = (A^{-1})^k$ for any integer k
- AB is non-singular and $(AB)^{-1} = B^{-1}A^{-1}$ *(notice order of matrices)*
- A^T is non-singular and $(A^T)^{-1} = (A^{-1})^T$

Inverses of 2x2 matrices:

Let $\delta = ad - bc$ *(the determinant)* then it is true that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -ab+ab \\ cd-dc & -cd+da \end{bmatrix} =$$

$$\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix} = \delta\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If $\delta \neq 0$ we can divide a 2x2 matrix by δ and easily find its inverse so that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } A^{-1} = \frac{1}{\delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $\delta = 0$ the A is singular.

Ex: Given $A = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$ find A^{-1} if possible.

Ex: Given $B = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ find B^{-1} if possible.

Inverses of Larger Matrices

If A is a nxn matrix to find A^{-1} if it exists we need to:

- Set up the augmented matrix $\left[A|I_n\right]$
- Reduce A to its row reduced echelon form
- If the LHS of the augmented matrix cannot be made into I_n, then A is singular.
- If the LHS of the augmented matrix becomes I_n, then the RHS is A^{-1}.

$$\left[A|I_n\right] \Rightarrow \left[I_n|A^{-1}\right]$$

there is a way to find the inverse for larger matrices using determinants, we will discuss this later.

Ex: Find the inverse if it exists for $A = \begin{bmatrix} 1 & -5 & 2 \\ 3 & 1 & -2 \\ 5 & 7 & -6 \end{bmatrix}$

Let A be a nxn matrix, then A is non-singular iff rank(A) = 0

Ex: Find the inverse if it exists for $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

Solving a System using the Inverse of a Coefficient Matrix

Suppose AX= B is a system with the same number of rows as variables. We know AX = B has a unique solution iff A is non-singular, then:

$$AX = B$$
$$A^{-1}(AX) = A^{-1}B$$
$$(A^{-1}A)X = A^{-1}B$$
$$I_nX = A^{-1}B$$
$$X = A^{-1}B$$

Ex: Solve the system $\begin{cases} -5x + 3y + 6z = 4 \\ 3x - y - 7z = 11 \\ -2x + y + 2z = 2 \end{cases}$
