Linear Algebra
Determinants and Eigenvalues

Introduction:

Many important geometric and algebraic properties of square matrices are associated with a single real number revealed by what's known as the determinant. For example the areas of volumes of certain geometric figures can be found by calculating the determinant of the matrix made up of the vectors that form these geometric figures.

We will also use the determinant to introduce a special value called the eigenvalue of the square matrix. The eigenvectors associated with the eigenvalue have the special property that when multiplied by the matrix they produce a parallel vector.

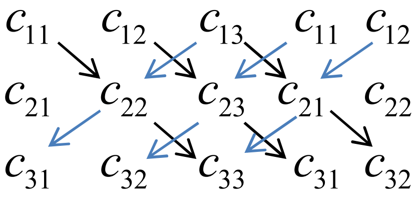
Introduction to Determinants

Any square nxn matrix A has a determinant. The notation is $|A|$ or det(A).

- a 1x1 matrix $A = [a_{11}]$ has a determinant determined by its only term det(A) = a_{11}
- a 2x2 matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a determinant determined by det(B) = ad – bc
- a 3x3 matrix $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$ has a determinant determined by

  det(C) = $c_{11}c_{22}c_{33} + c_{12}c_{23}c_{31} + c_{13}c_{21}c_{32} - c_{13}c_{22}c_{31} - c_{11}c_{23}c_{32} - c_{12}c_{21}c_{33}$

there is a pattern that makes this easy to memorize, its often called the "basket weaving technique".



Ex: Calculate the determinants of the following matrices

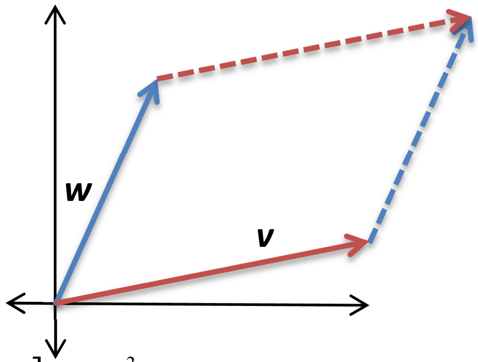
$$A = \begin{bmatrix} 7 & 5 \\ -8 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 4 & -6 \\ 3 & 5 & -4 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

Applications of Determinants: Areas and Volumes

Let $\vec{v} = [v_1, v_2]$ and $\vec{w} = [w_1, w_2]$ in $\Re^2$ that are not parallel and share a starting vertex. Then the area of the parallelogram formed by ***v*** and ***w*** if given by the determinant of

$$\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix}$$

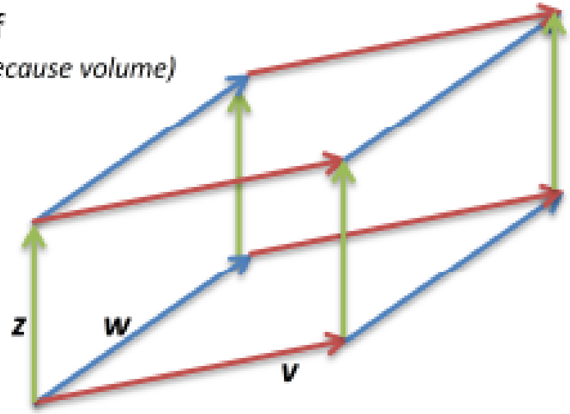
(take the absolute value since area should be non-negative)



Let $\vec{v} = [v_1, v_2, v_3]$ and $\vec{w} = [w_1, w_2, w_3]$ and $\vec{z} = [z_1, z_2, z_3]$ in $\Re^3$ that are not coplanar and they all share the same initial point. Then the volume of the parallelepiped form by ***v***, ***w***, and ***z*** is given by the determinant of

$$\begin{bmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

(again absolute value because volume)



What happens when the vectors are parallel or coplanar?

Calculating Determinants by Coplanar Expansion

There is another method of finding determinants using elements of a row multiplied by determinants of square matrices created by certain elements in the other rows.

Let A be a nxn matrix with n ≥ 2. The (i,j)th sub-matrix A_{ij} of the matrix A is the (n-1)x(n-1) matrix obtained by deleting the ith row of A and the jth column of A. The **(i,j)th minor**, denoted M_{ij}, is the determinant of the submatrix A_{ij} of A.

Let A be a nxn matrix with n ≥ 2. The (i,j)th cofactor of A, denoted C_{ij}, is $(-1)^{i+j}$ multiplied by the (i,j)th minor of A. In other words $C_{ij} = (-1)^{i+j} M_{ij}$

So given an 3x3 matrix there are nine submatrices, nine minors, and nine cofactors.

Ex: Find all nine submatrices, minors, and cofactors of $A = \begin{bmatrix} 2 & 5 & 6 \\ -1 & 4 & -7 \\ 3 & -2 & 9 \end{bmatrix}$

The Formal Definition of the Determinant

Let A be an nxn matrix. The determinant of A, denoted det(A), is defined as follows:

- If n = 1, then det(A) = a_{11}
- If n >1, then det(A) = $a_{n1}C_{n1} + a_{n2}C_{n2} + a_{n3}C_{n3} + \ldots + a_{nn}C_{nn}$

When we calculate the determinant using the sum of cofactors, we call the procedure the cofactor expansion.

Ex: Given $A = \begin{bmatrix} 2 & 5 & 6 \\ -1 & 4 & -7 \\ 3 & -2 & 9 \end{bmatrix}$ calculate the determinant by cofactor expansion 3 ways.

We could also define the determinant of a nxn matrix A with n >1 as

$$\det(A) = a_{1n}C_{1n} + a_{2n}C_{2n} + a_{3n}C_{3n} + \ldots + a_{nn}C_{nn}$$

Choose the row/column with the most zeros to simplify the computation since any of them with produce the determinant.

Ex: Calculate the determinant by cofactor expansion for the following

A. $A = \begin{bmatrix} 6 & 2 & 1 \\ -1 & 3 & -5 \\ 4 & 0 & 7 \end{bmatrix}$ **B.** $B = \begin{bmatrix} 3 & -1 & 2 & 4 \\ 6 & 5 & 1 & -5 \\ 3 & 0 & 2 & -4 \\ 7 & 0 & -3 & 9 \end{bmatrix}$

Determinants by Row Reduction

Determinant of Upper Triangular Matrices:

Let A be an upper triangular nxn matrix then, det(A) = $a_{11}a_{22}a_{33}\ldots.a_{nn}$ *(product of diagonal entries)*

Ex: calculate det(A) if $A = \begin{bmatrix} 3 & -1 & 4 & 5 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

What effect does row reduction have on the determinant?

Let A be a nxn matrix with the det(A)=P and let c be a scalar, then:

- If the row operation is $R_i \leftrightarrow R_j$ then the determinant becomes –P
- If the row operation if $R_i = cr_i$ the the determinant becomes cP
- If the row operation if $R_i = cr_j + r_i$ then the determinant stays the same

Ex: Use row reduction to calculate the determinant of $A = \begin{bmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & -1 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$

Theorem: A nxn matrix A is nonsingular iff det(A)≠0

Corollary: Let A be an nxn matrix, then rank(A)=0 iff det(A)≠0

Assume A is a nxn matrix, then the following are equivalent

A is singular (A^{-1} DNE)	**A is nonsingular (A^{-1} exists)**
rank(A)≠n	rank(A)=n
det(A)=0	det(A)≠0
AX = 0 has a nontrivial solution	AX = 0 has only the trivial solution
AX = B does not have a unique solution	AX = B has a unique solution

Further Properties of the Determinant

In this section we will discuss the determinant of a matrix product, the determinant of a transpose, the determinant and the inverse, and finish with Cramer's Rule to solve systems using only the determinants.

- If A and B are both nxn matrices then, det(AB) = det(A)*det(B)
- If A is a nxn matrix with A^{-1} as its inverse then, det(A) = $\frac{1}{\det(A^{-1})}$
- If A is an nxn matrix then, det(A)=det(A^T)

The Adjoint Matrix

Let A be a nxn matrix with n ≥ 2 then, the **adjoint**, denoted adj(A), **of A** is the nxn matrix whose ijth entry is the jith cofactor C_{ji} of A. The adjoint is the transpose of cofactors matrix. *Watch the order.*

$$adj(A) = \begin{bmatrix} C_{11} & C_{21}\cdots & C_{n1} \\ C_{12} & C_{22}\cdots & C_{n2} \\ C_{1n} & \cdots\cdot & C_{nn} \end{bmatrix}$$

Ex: Find the adjoint of $A = \begin{bmatrix} -4 & 0 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

If A is a nxn matrix with adjoint, adj(A) then:

$$A * adj(A) = adj(A) * A = \det(A) I_n$$

If a is a nxn invertible matrix, then $A^{-1} = \frac{1}{\det(A)} adj(A)$

Cramer's Rule

Cramer's Rule, named for Gabriel Cramer (1704-1752), uses determinants to a solve a system of n linear equations in n variables. This rule only applies to systems with unique solutions.

Cramer's Rule : Let AX = B represent a system of n linear equations in n variables such that det(A) ≠ 0 then the solution to the system is

$$x_1 = \frac{\det(A_1)}{\det(A)} \quad x_2 = \frac{\det(A_2)}{\det(A)} \quad \dots \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where the i^{th} column of A_i is the column of constants in the system of equations, *(B).*

Ex: solve by Cramer Rule $\begin{cases} 5x - 3y - 10z = -9 \\ 2x + 2y - 3z = 4 \\ -3x - y + 5z = -1 \end{cases}$

Eigenvalues and Diagonalization

We will define eigenvalues and eigenvectors for matrices in order to find when possible the diagonal form of a square matrix.

Let A be a nxn matrix. A real number **λ is an eigenvalue of A** iff there exists any nonzero vector X for which AX = λX. Also, any nonzero vector X for which AX = λX is called an eigenvector corresponding to the eigenvalue λ.

Let A be a nxn matrix and λ be an eigenvalue of A then, the set $E_\lambda = \{x | AX = \lambda X\}$ is called the **eigenspace** corresponding to the eigenvalue λ.

The Characteristic Polynomial of a Matrix A

We need procedure to determine all eigenvalues and eigenvectors of an nxn matrix A.

If X is an eigenvector for A corresponding to the eigenvalue λ then

$$AX = \lambda X = \lambda I_n X \text{ or } (\lambda I_n - A)X = 0$$

Therefore X is a nontrivial solution to the homogeneous system whose coefficient matrix is $(\lambda I_n - A)$.

Let A be a nxn matrix and λ be a real number then

- λ is an **eigenvalue** of A iff $\det(\lambda I_n - A) = 0$
- The **eigenvectors** corresponding to λ are the nontrivial solutions X to the homogeneous system $(\lambda I_n - A)X = 0$.
- The **eigenspace** E_λ corresponding to λ us the set of all nontrivial solutions to $(\lambda I_n - A)X = 0$

Let A be a nxn matrix then the **characteristic polynomial of A** is the polynomial given by

$$P_A(\lambda) = \det(\lambda I_n - A)$$

The real roots of the characteristic polynomial are the eigenvalues of A.

Procedure to find Eigenvalues, Eigenvectors, and Eigenspace:

1. State the characteristic polynomial by solving the determinant $P_A(\lambda) = \det(\lambda I_n - A)$
2. The values $\lambda_1, \lambda_2, \ldots \lambda_k$ that are the roots to $P_A(\lambda)$ are the eigenvalues
3. Set up the homogeneous system $(\lambda I_n - A)X = 0$ where the coefficient matrices are $(\lambda I_n - A)$ for each $\lambda_1, \lambda_2, \ldots \lambda_k$. The nontrivial solution(s) to this system is the eigenvector corresponding to each eigenvalue.
4. All linear combinations of the eigenvectors for each $\lambda_1, \lambda_2, \ldots \lambda_k$ make up the eigenspace corresponding to each eigenvalue and eigenvector. State $E_{\lambda k}$.

Ex: Find all the eigenvalues of the given matrix then state the eigenvectors associated with each eigenvalue and the corresponding eigenspace.

a. $A = \begin{bmatrix} 3 & 4 & 12 \\ 4 & -12 & 3 \\ 12 & 3 & -4 \end{bmatrix}$ **b**. $B = \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix}$

Diagonalization

We now seek to take a square matrix and turn it into a representative diagonal matrix by use of eigenvalues and eigenvectors. Realize if we can replace a matrix with a diagonal matrix then we will greatly simplify any computations involving the original matrix. Therefore our next goal is to present a formal method for using eigenvalues and eigenvectors to find a diagonal form of a square matrix, provided it exists.

A matrix A' is **similar** to a matrix A if there exists some nonsingular matrix P such that $A' = P^{-1}AP$.

Properties of Similar Matrices

Let A, B and C be nxn matrices. then the following properties are true.

1. A is similar to A
2. If A is similar to B, then B is similar to A.
3. If A is similar to B and B is similar to C then A is similar to C.
4. If A and B are similar they have the same eigenvalues.

An nxn matrix A is **diagonalizable** when A is similar to a diagonal matrix. That is, A is diagonalizable when there exists an invertible matrix P such that $D = P^{-1}AP$ where D is a diagonal matrix.

An nxn matrix A is diagonalizable iff it has n linearly independent eigenvectors.

Let A and P be nxn matrices such that each column of P is an eigenvector of A. If P is nonsingular then $D = P^{-1}AP$ is a diagonal matrix similar to the matrix A. The i^{th} main diagonal entry d_{ii} of D is the eigenvalue for the eigenvector forming the i^{th} column of P.

Notice in the two examples we did for finding eigenvalues and eigenvectors, we see if we form P in either case, then P^{-1} doesn't exist and hence, the original matrices are not diagonalizable.

Method for Diagonalizing a nxn matrix A (if possible)

1. Calculate the characteristic polynomial and find all the real roots, $\lambda_1, \lambda_2, \ldots \lambda_k$. In other words find all λ such that if $P(\lambda) = \det(\lambda I_n - A)$ then $P(\lambda) = 0$.
2. For each eigenvalue λ_m in $\lambda_1, \lambda_2, \ldots \lambda_k$ we find the corresponding eigenvectors. If there are n λ's that are distinct, then A is diagonalizable and if there are less than n λ's, then A is not diagonalizable.
3. If there are n λ's that are distinct then there are n eigenvectors. Form a matrix P such that the columns of P are eigenvectors and find P^{-1}.
4. Check your work by checking that $P^{-1}AP$ is a diagonal matrix D whose main diagonal entries are the corresponding eigenvalues for the eigenvectors that formed that column.

Ex: Find the diagonal matrix D such that $D = P^{-1}AP$ for $A = \begin{bmatrix} -4 & 8 & -12 \\ 6 & -6 & 12 \\ 6 & -8 & 14 \end{bmatrix}$