Limits and Their Properties

A Preview of Calculus
Objectives: Understand what calculus is and how it compares with precalculus.Understand that the tangent line problem is basic to calculus. Understand that the area problem is also basic to calculus.

Calculus is the mathematics of change – velocities and accelerations. Calculus is also the mathematics of tangent lines, slopes, areas, volumes, arc length, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

Fundamental Differences between Pre-Calc and Calc

- An object traveling at a constant velocity can be analyzed with precalculus. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with precalculus. To analyze the slope of a curve, you need calculus.
- A tangent line to a circle can be analyzed with precalc. To analyze a tangent line to a general graph you need calc.
- The area of a rectangle can be analyzed with precalc. To analyze the area under a general curve, you need calc.

The main way we switch from precalc. to calc. is the use of a limit process. Calculus is a "limit machine".

Two main areas we will discuss in Calc I are:
1. The Tangent line problem
2. The Area Problem

Finding Limits Graphically and Numerically

Objective: Estimate a limit using a numerical or graphical approach. Learn different ways that a limit can fail to exist. Study and use a formal definition of limit.

An Intro to Limits

Sketch to graph of

$$f(x)=\frac{x^3-1}{x-1},\ x\neq 1$$

The graph is a parabola with a hole at (1,3)

Although x can not equal 1 for this function you can see what happens to f(x) as x approaches 1 from **both directions**. The notation used is

$$\lim_{x\to c} f(x)= \quad or \quad \lim_{x\to 1} f(x)=$$

This table shows us what is happing in the graph as well as the limit

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
f(x)	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

Then we can say $\lim_{x \to 1} f(x) = 3$

The limit must be the same from both directions!!!

<u>Three pronged approach to problem solving (finding limits)</u>

1. Numerical approach – Construct a table of values
2. Graphical approach – Draw a graph by hand or using technology
3. Analytic approach – Use algebra or calculus

Ex: Find the limit of f(x) as x approaches 3 where f is defined as

$$f(x) = \begin{cases} 1, & x \neq 3 \\ -1, & x = 3 \end{cases}$$

<u>Common Types of Behavior Associated with Nonexistence of a Limit:</u>

1. f(x) approaches a different number from the right side of c that is approaches from the left side.
2. f(x) increases or decreases without bound as x approaches c.
3. f(x) oscillates between two fixed values as x approaches c.

Some examples of limits that fail to exist

a. $\lim_{x \to 0} \frac{|x|}{x}$ **b.** $\lim_{x \to 0} \frac{1}{x^2}$ **c.** $\lim_{x \to 0} \sin \frac{1}{x}$

<u>A Formal Definition of Limit</u>

Let f be a function defined on an interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \quad then \quad |f(x) - L| < \varepsilon$$

Ex: Use the $\varepsilon - \delta$ definition of a limit to prove that $\lim_{x\to 2}(3x-2)=4$

Solution: You must show that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|(3x-2)-4| < \varepsilon$ whenever $0 < |x-2| < \delta$

you should try to work with $|f(x)-L|$ *and simplify it to* $|x-c|$ *that will let you see what* δ *should equal in terms of* ε

This is how the proof should be written formally:

Prove: $\lim_{x\to 2}(3x-2)=4$

Given ε let $\delta = \varepsilon/3$ then

$$
\begin{aligned}
|x-c| < \delta &= |x-2| < \varepsilon/3 \\
&= 3|x-2| < \varepsilon \\
&= |3x-6| < \varepsilon \\
&= |(3x-2)-4| < \varepsilon \\
&= |f(x)-L| < \varepsilon
\end{aligned}
$$

Q.E.D.

Evaluating Limits Analytically

Objective: Evaluate a limit using properties of limits. Develop and use a strategy for finding limits. Evaluate a limit using dividing out and rationalizing techniques. Evaluate a limit using the Squeeze Theorem.

Some Basic Limits:
Let *b* and *c* be real numbers and let *n* be a positive integer.

1. $\lim_{x\to c} b = b$ **2.** $\lim_{x\to c} x = c$

Ex: **a.** $\lim_{x\to 6}(-2)$ **b.** $\lim_{x\to 7} x$

Properties of Limits:
Let *b* and *c* be real numbers, let *n* be a positive integer, and let *f* and *g* be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K$$

1. Scalar multiple: $\lim_{x\to c}\left[bf(x)\right] = bL$

2. Sum or difference: $\lim_{x\to c}\left[f(x) \pm g(x)\right] = L \pm K$

3. Product: $\lim_{x\to c}\left[f(x)g(x)\right] = LK$

4. Quotient: $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ provided $K \neq 0$

5. Power: $\lim_{x\to c}\left[f(x)\right]^n = L^n$

Ex: a. $\lim_{x\to 3}(5x^2 - 3x + 2)(x - 1)$ b. $\lim_{x\to -2}\left(\frac{x^2+5}{x-2}\right)^2$

Limits of Polynomials and Rational Functions:
If p(x) and q(x) are polynomials and c is a real number, then

$$\lim_{x\to c} p(x) = p(c) \quad \text{and} \quad \lim_{x\to c}\frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \quad \textit{if} \quad q(c) \neq 0$$

The Limit of a Function Containing a Radical:
Let *n* be a positive integer. The following limit is valid for all *c* if *n* is odd, and is valid for *c* > 0 if *n* is even.

$$\lim_{x\to c}\sqrt[n]{x} = \sqrt[n]{c}$$

Limits of Trigonometric Functions
Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x\to c}\sin x = \sin c$ **2.** $\lim_{x\to c}\cos x = \cos c$

3. $\lim_{x\to c}\tan x = \tan c$ **4.** $\lim_{x\to c}\cot x = \cot c$

5. $\lim_{x\to c}\sec x = \sec c$ **6.** $\lim_{x\to c}\csc x = \csc c$

Ex: a. $\lim_{x \to \frac{\pi}{4}} \sec x =$ **b.** $\lim_{x \to \pi} (x \cos x) =$ **c.** $\lim_{x \to 0} \sin^2 x =$

Functions That Agree at All But One Point:
Let c be a real number and let f(x) = g(x) for all x≠c in an open interval containing c. If the limit of g(x) as x approaches c exists, then the limit of f(x) also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

Ex: a. $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$ **b.** $\lim_{x \to 2} \frac{2 - x}{x^2 - 4}$

<u>A Strategy for Finding Limits:</u>

1. Learn to recognize which limits can be evaluated by direct substitution
2. If the limit of f(x) as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than x = c .
3. Apply the above theorem to conclude analytically that

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = g(c)$$

4. Use a graph or table to reinforce your conclusion

Ex: a. $\lim_{x \to -4} \frac{x^2 + x - 6}{x - 4}$ **b.** $\lim_{x \to 2} \frac{\sqrt{x+2} - 2}{x - 2}$

<u>The Squeeze Theorem:</u>
for h(x) ≤ f(x) ≤ g(x) for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

then $\lim_{x \to c} f(x)$ exists and is equal to L.

<u>Two Special Trigonometric Limits:</u>

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$ **2.** $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

Ex: a. $\lim_{x\to 0}\frac{\tan x}{5x}$ **b.** $\lim_{x\to 0}\frac{\sin 4x}{x}$ **c.** $\lim_{x\to\frac{\pi}{4}}\frac{1-\tan x}{\sin x-\cos x}$

Continuity and One-Sided Limits:

Objective: Determine continuity at a point and continuity on an open interval. Determine one-sided limits and continuity on a closed interval. Use properties of continuity. Understand and use the Intermediate Value Theorem.

A **continuous function** is one whose graph can be drawn without a "pen" leaving the paper.

Continuity at a Point and on an Open Interval

Continuity at x = c can be destroyed by any one of the following conditions

1. The function is not defined at x = c.
2. The limit of f(x) does not exist at x = c.
3. The limit of f(x) exists at x = c, but it is not equal to f(c).

Definition of Continuity:

Continuity at a Point: A function f is **continuous at c** if the following three conditions are met.

1. f(c) is defined
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

Ex: Show that $f(x)=3x+2$ is continuous at x = 2

Continuity on an Open Interval: A function is **continuous on an open interval (a,b)** if it is continuous at each point in the interval . A function that is continuous on the entire real line (-∞,∞) is **everywhere continuous.**

If f is not continuous at x = c then f is said to have a discontinuity at c. Discontinuities fall into 2 categories: **Removable** and **Unremovable.**

Look at the following:

a. $f(x)=\frac{1}{x}$ **b.** $g(x)=\frac{x^2-1}{x-1}$ **c.** $h(x)=\begin{cases}x+1, & x\le 0\\ x^2+1, & x>0\end{cases}$ **d.** $y=\sin x$

One Sided Limits and Continuity on a Closed Interval

If *f*(x) approaches L as *x* tends toward *c* from the left ($x < c$), we write

$$\lim_{x \to c^-} f(x) = L$$ *limit from the left*

Likewise, if f(x) approaches M as *x* tends toward *c* from the right ($c < x$), then

$$\lim_{x \to c^+} f(x) = M$$ *limit from the right*

Ex: a. $\lim_{x \to -2^+} \sqrt{4 - x^2}$ **b.** $\lim_{x \to 0^+} [x]$

Existence of a Limit *(Alternative Definition)*

The two sided limit $\lim_{x \to c} f(x)$ exists iff the two one-sided limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and are equal, Therefore:

$$\lim_{x \to c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

Continuity on a Closed Interval

A function *f* is continuous on the closed interval [a,b] if it is continuous on the open interval (a,b) and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b)$$

Ex: Look at continuity of $f(x) = \sqrt{1 - x^2}$

Infinite Limits

Objective: Determine infinite limits from the left and from the right. Find and sketch the vertical asymptotes of the graph of a function.

Vertical Asymptotes *(Definition)*

If ***f(x)*** approaches infinity (or negative infinity) as ***x*** approaches ***c*** from the right or left, then ***x = c*** is a vertical asymptote of the graph of ***f.***

Vertical Asymptote: *(Theorem)*

Let ***f*** and ***g*** be continuous on an open interval containing ***c***. If ***f(c) ≠ 0***, ***g(c) = 0***, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at ***x = c***.

Ex: Determine all vertical asymptotes of the graph

a. $f(x)=\dfrac{1}{2(x+1)}$ **b.** $g(x)=\dfrac{x^2+1}{x^2-1}$ **c.** $h(x)=\dfrac{x^2+2x-8}{x^2-4}$

Let f be a function given by

$$f(x)=\frac{3}{x-2}$$

from the graph you can see the f(x) decreases without bound as x approaches 2 from the left, and f(x) increases without bound as x approaches 2 from the right. This behavior is denoted by:

$$\lim_{x\to 2^-} f(x)=-\infty \text{ and } \lim_{x\to 2^+} f(x)=\infty$$

Also can be seen in a table

x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
f(x)	-6	-30	-300	-3000	?	3000	300	30	6

A limit in which f(x) increases or decreases without bound as x approaches c is called an **infinite limit.**

Remember the statement ***lim f(x) = ∞ does not mean the limit exists!*** *It actually tells us how the limit fails to exist by denoting the unbounded behavior of f(x) as x approaches c.*

Ex: a. $\lim\limits_{x\to 1^-}\dfrac{1}{x-1}$ **b.** $\lim\limits_{x\to 1}\dfrac{1}{(x-1)^2}$ **c.** $\lim\limits_{x\to 3}\dfrac{2x-5}{(x-3)^4}$

Properties of Infinite Limits:

Let c and L be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x)=\infty \text{ and } \lim_{x\to c} g(x)=L$$

1. Sum or Difference: $\lim\limits_{x\to c}\left[f(x)\pm g(x)\right]=\infty$

2. Product:
$$\lim_{x\to c}\left[f(x)g(x)\right]=\infty,\ L>0$$
$$\lim_{x\to c}\left[f(x)g(x)\right]=-\infty,\ L<0$$

3. Quotient: $\lim\limits_{x\to c}\dfrac{g(x)}{f(x)}=0$

Similar properties hold for one – sided limits and for functions for which the limit of f(x) as x approaches c is -∞

Ex: a. $\lim_{x\to 0}\left(1+\frac{1}{x^2}\right)$ **b.** $\lim_{x\to 0^+} -3\cot x$

Limits at Infinity

Objective: Determine (finite) limits at infinity. Determine the horizontal asymptotes, if any, of the graph of a function. Determine infinite limits at infinity.

Look at the end behavior of $f(x)=\frac{3x^2}{x^2+1}$

Graphically you will see that as x increases or decreases without bound f(x) approaches 3.

Also look at a table:

x	$-\infty\leftarrow$	-100	-10	-1	0	1	10	100	$\rightarrow\infty$
f(x)	$3\leftarrow$	2.9997	2.97	1.5	0	1.5	2.97	2.9997	$\rightarrow 3$

So we write

$\lim_{x\to\infty} f(x)=3$ and $\lim_{x\to-\infty} f(x)=3$

Definition of a Horizontal Asymptote:

The line *y* = *L* is a **horizontal asymptote** of the graph of ***f(x)*** if as

$x\to-\infty$ or $x\to\infty$ then $f(x)\to L$

or

The line y = L is a **horizontal asymptote** of the graph of f if

$$\lim_{x\to-\infty} f(x)=L \;\; or \;\; \lim_{x\to\infty} f(x)=L$$

Limits at Infinity

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty}\frac{c}{x^r}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{c}{x^r}=0$$

Ex: Find the limit: $\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right)$

<u>How to Find Limits at ±∞ of Rational Functions:</u>

1. If the degree of the numerator is less than the degree of the denominator, then the limit of the rational function is 0.

2. If the degree of the numerator is equal to the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.

3. If the degree of the numerator is greater than the degree of the denominator, then the limit of the rational function does not exist, but you may still be able to give an answer of ∞ or -∞.

Ex: Find each limit

a. $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$ **b.** $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$ **c.** $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$

Ex: Find each limit:

a. $\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ **b.** $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ **c.** $\lim_{x\to\infty} \sin x$