Functions and Their Graphs

Functions:

Section Objectives: Students will know how to use function notation and how to evaluate functions and find their domains.

A **relation** is a set of ordered pairs.
The **domain** of a relation is the set of all *x*-coordinates of the relation.
The **range** of a relation is the set of all *y*-coordinates of the relation.

A **function**, **(f)**, from a set A to a set B is a relation that assigns to each element *x* in the set A to exactly one element *y* in set B. The set A is the domain (inputs) of *f* and the set B contains the range (outputs).

Ex: $S = \{(1, 3), (-5, 4), (7, 3)\}$
a) Is this a relation? Function?
b) What is the domain and range of *S*?

Characteristics of a function from set A to set B

1. Each element in A must be matched with an element in B.
2. Some elements in B may not be matched with any element in A.
3. Two or more elements in A may be matched with the same element in B.
4. An element in A (the domain) cannot be matched with two different elements in B.

For functions the variable representing the domain elements (*x*) is called the **independent variable.** The variable representing the range elements (*y*) is called the **dependent variable.** An equation in two variables (*x* and *y)* represents *y* as a function of *x* if it can be uniquely solved for *x*.

Ex: Which of the following equations represent *y* as a function of *x*?
a) $2x^2 + y + 1 = 0$ **b)** $y^2 = x$

Function Notation:

We *name* a function so that it can be referenced. Typically we use the name *f*. Since we say *y* is a function of *x*, replace *y* with *f*(*x*). This value is read “*f* of *x*.”

Tip: There are two concepts that we cannot emphasize too much. One is that y = f(x). The other is that f is the name of the function, not a variable.

Ex:. Given f(x) = $x^2 - 4x$, find the following.

a) $f(2)$ **b)** $f(4a)$ **c)** $f(x - 2)$ **d)** $f(x + h)$

Ex: Given f(x) = $-x^2 + 4x + 1$

a) $f(2)$ **b)** $f(t)$ **c)** $f(x - 2)$ **d)** $f(x + h)$

Ex: Evaluate when x = -1, 0, 1

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \ge 0 \end{cases}$$

The Domain of a Function

The implied domain of a function is the set of all x such that the corresponding y is a real number. At least for a while, we will only consider three situations:

1. Polynomials: domain is $(-\infty, \infty)$.

2. Fractions: x cannot be any number in the domain that makes the denominator zero.

3. Radicals: if the index is even, then the radicand must be nonnegative.

Ex: Find the domains of the following functions.

a) f(x) = $x^3 + 3x + 1$ **b)** $f(x) = \frac{1}{x^2 - 1}$ **c)** $f(x) = \sqrt{x - 2}$

The Graph of a Function

Since *y* = *f*(*x*), graphing functions is no different from graphing equations in two variables.

The **graph of a function** *f* is the collection of ordered pairs (*x, f (x)*) such that *x* is in the domain of *f*.

Vertical Line Test: If every vertical line meets the graph of a relation in at most one point, then the relation is a function.

Draw the graph of $\boldsymbol{y^2 = x}$,is the graph of a function?

Ex: Which of the following are graphs of functions?

a) $y = -x^2$ **b)** $x^2 + y^2 = 4$

Applications of functions

One important application of equations and functions is the average and instantaneous rates of change, this is a building block of calculus.

If (x_1, y_1) and (x_2, y_2) are points on a function then the change in y is $y_2 - y_1$ and the change in x is $x_2 - x_1$. Therefore, the **average rate of change** of the function is found by

$$\frac{y_2 - y_1}{x_2 - x_1} \text{ or } \frac{\Delta y}{\Delta x}$$

Ex: Find the average rate of change for $f(x) = x^2 + 3$ on the interval [1,3]

Another application of functions related to rate of change from calculus called the **difference quotient**.

$$DQ = \frac{f(x+h) - f(x)}{h}$$

Tip: This is one of the single most important things you will need in calculus, make sure you know it forwards and backwards.

Ex: Find the Difference Quotient for

a) $f(x) = 3x + 2$

b) $g(x) = x^2 + 2x - 1$

c) $h(x) = \sqrt{x}$

Tip: *try to eliminate the h in the denominator. In calculus we see what happens when* $h \to 0$

The difference quotient can also be written as $DQ = \dfrac{f(x + \Delta x) - f(x)}{\Delta x}$