Finite Dimensional Vector Spaces

We will define vector spaces to be algebraic structures with operation having properties similar to addition and scalar multiplication in $\Re^n$ and **M**$_{mn}$ (the set of all m×n matrices).

Intro to Vector Spaces

A **vector space** is a set **V** on which two operations vector addition and scalar multiplication are defined, and the following properties hold true for every vector $\vec{v} \in \mathrm{V}$, $\vec{w} \in \mathrm{V}$, and $\vec{z} \in \mathrm{V}$ and for every $c \in \Re$ and $k \in \Re$:

1. $\vec{v} + \vec{w}$ is in **V** *closure property under addition*
2. $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ *commutative property for vector addition*
3. $(\vec{v} + \vec{w}) + \vec{z} = \vec{v} + (\vec{w} + \vec{z})$ *associative property for vector addition*
4. $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$ *additive identity property, which is zero*
5. $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$ *additive inverse property which is –v*
6. $c\vec{v}$ is in **V** *closure property under scalar multiplication*
7. $\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w}$ *distributive law for scalars*
8. $(\lambda + k)\vec{v} = \lambda\vec{v} + k\vec{v}$ *distributive law for scalars*
9. $(\lambda k)\vec{v} = \lambda(k\vec{v}) = k(\lambda\vec{v})$ *associative property for scalar multiplication*
10. $1(\vec{v}) = \vec{v}(1) = \vec{v}$ *multiplicative identity property for scalar multiplication, which is 1*

It's important to realize that a vector space consists of four entities: a set of vectors, a set of scalars, and two operations. Even further **ANY** set that satisfies these properties is called a vector space and the objects in the set are called vectors.

Ex: Show that the set of ordered pairs in R^2 with the standard operations if a vector space.

Note:

1. In any vector space, the additive identity element and the additive inverse element are unique.
2. $V \in \Re^n$ and $V \in M_{mn}$ are vector spaces under the usual operations of vector addition and scalar multiplication.
3. Some vector spaces have additional properties defined on them as well, for example:
   - $\Re^n$ has a dot product
   - M_{mn} has matrix multiplication, the transpose, and the cross product
4. There is no such thing, in general, as vector multiplication of vector division. Thus in general, the only operation that combines two vectors is vector addition.

Ex: Show that the set of all 2 x 3 matrices with the operations of matrix addition and scalar multiplication is a vector space.

Ex: Show that the set of all degree two or less polynomials is a vector space. Let addition and scalar multiplication be defined in the "usual" way.

Ex: Show that the set of all integers, Z is **NOT** a vector space

Some Important Vector Spaces:

- $\Re$ = the set of all real numbers
- $\Re^2$ = the set of all ordered pairs
- $\Re^3$ = the set of all ordered triples
- $\Re^n$ = the set of all n-tuples
- C(-∞,∞) = the set of all continuous functions defined on the real number line
- C[a,b] = the set of all continuous functions defined on the real number line
- P = the set of all polynomials
- P_n = the set of all polynomials of degree ≤ n
- M_{mn} = the set of al mxn matrices
- M_{nn} = the set of all nxn square matrices

Subspaces

Set A is a **subset** of set B iff each element of set A is an element of set B

Let **V** be a vector space, then **W is a subspace of V** iff **W** is a subset of **V** and **W** is itself a vector space with the same operations as **V**.

Given a vector space V, then:

- **V** is a subspace of itself
- The subset $\{\bar{0}\}$ under the same operation as **V** is called a subspace of **V**

All subspaces of **V** other than itself are called proper subspaces of **V**. The subspace $\{\bar{0}\}$ is called the trivial subspace of **V**.

Facts:

1. A vector space containing at least one nonzero vector has at least two subspaces:
   - The trivial subspace
   - The vector space itself
2. The real numbers have only two subspaces
3. All subspaces of R^n resemble one of the following: $\{\bar{0}\}$, R, R^2, R^3…,R^n

Test for a Subspace:

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following closure conditions hold:

1. If **u** and **v** are in W, then **u + v** is in W.

2. If **u** is in W and c is any scalar, then c**u** is in W.

Ex: Show that the set $W = \{(0, x_2, x_3) : x_2 \ and \ x_3 \ are \ real\}$ is a subspace of R^3 with the standard operations.

Ex: Show that **W** = the set of all 2 x 2 symmetric matrices is a subspace of the vector space $M_{2,2}$ with the standard operations of matrix addition and scalar multiplication.

When is a subset a subspace?
Not every subset of a vector space is a subspace. A subset **S** of a vector space **V** fails to be a subspace of **V** if **S** does not satisfy the properties of a vector space in its own right or if **S** does not use the same operation as **V**.

<u>The Span:</u>
The set of all linear combinations of the vectors is a subset **S** of **V** forms an independent subspace of **V** called **<u>the span</u> of S in V.**

Let **S** be a nonempty subset of a vector space **V**. Then a vector **v** is a linear combination of the vectors in **S** iff there exists a finite subset $\{v_1, v_2, \ldots. v_n\}$ of S such that $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_n\vec{v}_n$ for some real numbers $a_1, a_2, \ldots, a_n$.

Ex: The vector [10,-3,7] is a linear combination of the vectors is
S = {[2,-1,1], [1,0,1], [-2,-1,-1]} as seen by $[10,-3,7] = 3[2,-1,1] + 4[1,0,1] + 0[-2,-1,-1]$

Note: we can always use all the vectors in S by placing a zero in front of the ones we don't want.

Ex: Find all the linear combinations of **S** = {[3,4,1]}.

Let **S** be a nonempty subset of a vector space **V**, then the span of **S** in **V** is the set of all possible finite linear combinations of the vectors is **S**, notation: span(**S**)={all linear combinations of the vectors in **S**}.

Let **S** be a nonempty subset of a vector space **V**. then the following are true:

- **S** is a subset of the span(**S**) : $S \subseteq span(S)$
- span(**S**) is a subspace of V
- If W is a subspace of **V** with $S \subseteq W$, then span(**S**) $\subseteq$ **W**.
- The span(**S**) is the smallest subspace of **V** containing **S**.

The above theorem asserts that span(**S**) is created by adding to **S** precisely those vectors needed to make the closure properties hold true. Thus the whole idea of span is the "close up" a subset of a vector space to create a subspace.

Ex: Let v_1 and v_2 be any two vectors in R^4 then span($\{v_1,v_2\}$) is the smallest possible subspace of R^4 containing v_1 and v_2. If v_1 = [1,3,-2,5] and v_2 = [0,-4,3,-1] find span($\{v_1,v_2\}$).

The row space of a matrix is the span of the set **S** consisting of all the vectors in the rows of the matrix: row space(**A**) = span(**S**).

Calculating the span by the row space technique:
For finite sets of vectors in R^n we can find span(S) by forming the matrix A whose rows are the vectors in S. Then span(S) is the row space of A.

The span of the empty set is = $\{\vec{0}\}$

Ex: Find the span of $S = \{[1,0,-1],[0,1,1]\}$

Ex: Find the span of $T = \{[1,-1,1],[2,-3,3],[0,1,-1]\}$

Linear Independence
A set of vectors $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, ..., \vec{v}_k\}$ in a vector space V is called **linearly independent** when the vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_k\vec{v}_k = 0$ has only the trivial solution for $c_1, c_2, ... c_k$.
If there are also nontrivial solutions, then S is called **linearly dependent**.

Ex: Is the set $S = \{[3,2,4],[39,26,52]\}$ in R^3 is linearly dependent or independent.

Method to Check Linear Independence
Let $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, ..., \vec{v}_k\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent or dependent, use the following:

1. From the vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_k\vec{v}_k = 0$ write a system of linear equations in the variables $c_1, c_2, ... c_k$.
2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution then the set S is linearly independent. If the system also has nontrivial solutions, then S is linearly dependent.

Ex: Determine whether the following are linearly independent of dependent:
a. $S = \{[3,1,-1],[-5,-2,2],[2,2,-1]\}$ **b.** $S = \{1 + x - 2x^2, 2 + 5x - x^2, x + x^2\}$

Basis and Dimension
Suppose **S** is a subset of a vector space **V** and $\vec{v} \in V$. The following questions arise:

1. Is there a linear combination of vectors is S that equal $\vec{v}$?
2. If yes, is it the only such linear combination?

To answer the first question is equivalent to determining whether or not $\vec{v}$ is in span(**S**). To answer the second question is equivalent to determining whether or not the set **S** is linearly independent.

Let V be a vector space and B be a subset of V. Then **B is <u>a basis</u> for V** if and only iff:

i. B spans V
ii. B is linearly independent

Ex: Is $S = \{[1,1,1,1],[1,1,1,-1],[1,1,-1,-1],[1,-1,-1,-1]\}$ a basis for R^4?

Linear Transformations

We now start to look at functions whose domain and codomain are vector spaces. In other words functions that map vectors is one vector space to vectors in another vectors space. These are called transformations, we will look at a special case called **linear transformations.**

Let V and W be vector spaces and let f be a function form V to W ($f : V \rightarrow W$). So for all $\vec{v} \in V$, $f(\vec{v}) \in W$ and each $\vec{v}$ gives exactly one $f(\vec{v})$. The function f is **<u>a linear transformation</u>** iff :

i. $f(\vec{v}_1 + \vec{v}_2) = f(\vec{v}_1) + f(\vec{v}_2)$, $\forall \vec{v}_1, \vec{v}_2 \in V$
ii. $f(k\vec{v}_1) = kf(\vec{v}_1)$, $\forall k \in \Re$, $\forall \vec{v}_1 \in V$

Ex: Show that the mapping $f : M_{mn} \rightarrow M_{nm}$ defined by $f(A) = A^T$ for any m x n matrix A, is a linear transformation.

Ex: Show that the mapping $g : P_n \rightarrow P_{n-1}$ defined by $g(\vec{p}) = \vec{p}'$ is a linear transformation.

Note: Not every function that maps a vectors space into a vector space will be a linear transformation.

Ex: Show that $h : \Re^2 \rightarrow \Re^2$ defined by $h([x, y]) = [x-5, y+3]$ is not a linear transformation.

Let V be any vector space. Any linear transformation of the form f: V → V is called a **<u>linear operator</u>**. For this the domain and codomain of the transformation are the same.

Let L: V → W be a linear transformation, then the following are true:

- If V\` is a subspace of V, then the image of V\` in W is a subspace of W (the range of L is a subspace of W)
- If W\` is a subspace of W, then the pre-image of W\` in V is a subspace of V

The Matrix of a Linear Transformation

Here we will see the behavior of any linear transformation L: V →W is determined by the effect on a basis for V.

Facts:

1. Multiplication by an m x n matrix is always a linear transformation from R^n to R^n.
2. If the action of a linear transformation L: V →W in a basis for V is known then the action of L can be computed for all elements of V.

3. Every linear transformation of the form L: $R^n \rightarrow R^m$ can also be described by the product of some matrix and the vectors in R^n.

Consider a linear transformation T:$R^3 \rightarrow R^3$ which is a better representation:

$T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$ or $T(\mathrm{x}) = A\mathrm{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The second is easier for three reasons: it's easier to write, read, and program.

The key to representing a linear transformation L: V →W by a matrix is to determine how it acts on a basis for V.

We will define the standard basis for Rn to be $B = \{e_1, e_2, ..., e_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, ..., \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

Standard Matrix for a Linear Transformation:

Let T:$R^n \rightarrow R^m$ be a linear transformation such that, for the standard basis vectors e_i of R^n,

$$T(\mathrm{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, T(\mathrm{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, ..., T(\mathrm{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

then the m x n matrix whose columns correspond to T(e_i)

$$A = \begin{bmatrix} a11 & a12 & \dots & a1n \\ a21 & a22 & \dots & a2n \\ \vdots & \vdots & & \vdots \\ am1 & am2 & \dots & amn \end{bmatrix}$$

is such that T(**v**) = A**v** for every **v** in R^n. A is called the **standard matrix** for T.

Ex: Find the standard matrix for the linear transformation T:$R^3 \rightarrow R^2$ defined by $T(x, y, z) = (x - 2y, 2x + y)$

Ex: Find the standard matrix for the linear transformation T:$R^2 \rightarrow R^3$ defined by $T(x, y) = (2x - 3y, x - y, y - 4x)$