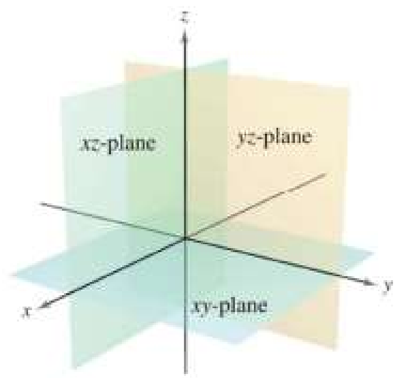
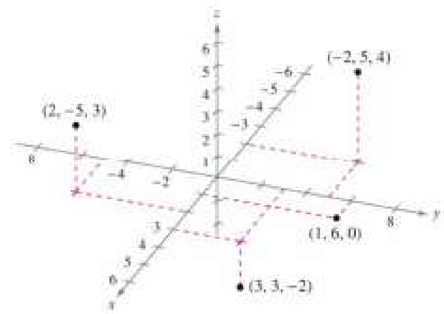
Vectors and the Geometry of Space

Vector Space

The 3-D coordinate system (rectangular coordinates) is the intersection of three perpendicular (orthogonal) lines called coordinate axis: x, y, and z. Their intersection is the origin. Therefore any point in space has three coordinates (x, y, z). The coordinate axis breaks the space into eight octants.





Recall:

- Any point in 2-D space creates a unique rectangle with the coordinate axis
- Every point (a, b) has a 1-to-1 correspondence with a point graphed in the plane.; denote R^2

Facts:

- Any point in 3-D space creates a unique rectangular prism with the coordinate planes
- Every point (a, b, c) has a 1-to-1 correspondence with a point graphed in space; denoted R^3

Distance Formula in 3-D

Given any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ the distance between them is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \|\overline{P_1P_2}\|$$

Midpoint Formula

The midpoint between two points P_1 and P_2 is

$$\text{Midpoint} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}, \frac{z_2+z_1}{2}\right)$$

Ex: Find the distance and midpoint between (5, -3, 7) and (2, -1, 6)

Equation of a sphere

The standard equation of a sphere with a center of (x_1, y_1, z_1) and a radius r is

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$$

Ex: Characterize the equation $x^2 + y^2 + z^2 + 8x + 10y - 6z + 41 = 0$

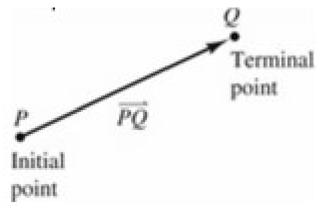
Vectors in the Plane

Vector: A directed line segment that has magnitude (length) and direction.

Notation: $\vec{v} = \mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ where $v_1, v_2, \ldots, v_n$ are called the components of the vector

or

$\vec{v} = \boldsymbol{v} = \overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2, \ldots, q_n - p_n \rangle$, where $P(p_1, p_2, \ldots, p_n)$ is the initial point and $Q(q_1, q_2, \ldots, q_n)$ is the terminal point.



(brackets can also be used to denote vectors)

Note: $\overrightarrow{QP} = \langle p_1 - q, p_2 - q_2, \ldots, p_n - q_n \rangle$

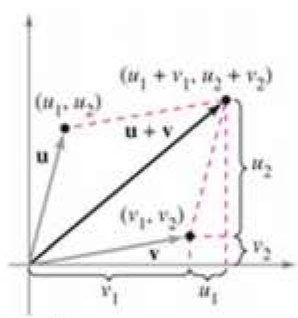
zero vector: the zero vector is the unique vector denoted $\vec{0}$ that has no magnitude and no direction

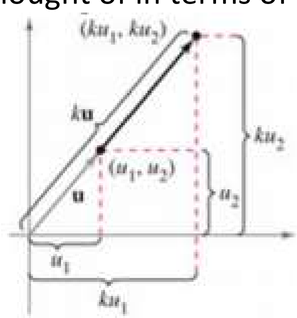
magnitude of a vector: given a vector $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ in component form, the magnitude is denoted $\|\vec{v}\|$ is given by $\|\vec{v}\| = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_n}^2}$. Given a vector $\overrightarrow{PQ}$ with initial point $P(p_1, p_2, \ldots, p_n)$ and terminal point $Q(q_1, q_2, \ldots, q_n)$ then $\|\overrightarrow{PQ}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + \cdots + (q_n - p_n)^2}$

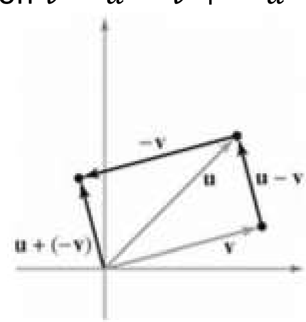
Vector Operations:

Let $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and let *k* be a scalar

- The vector sum of $\vec{v} + \vec{u} = \langle v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n \rangle$
- The scalar multiple of $\vec{v}$ and k is $k\vec{v} = \langle kv_1, kv_2, \ldots, kv_n \rangle$
- The negative of the vector is $-\vec{v} = \langle -v_1, -v_2, \ldots, -v_n \rangle$, $-\vec{v}$ goes opposite direction
- The vector difference can be thought of in terms of addition $\vec{v} - \vec{u} = \vec{v} + -\vec{u}$







Properties of Vectors:

Given the vectors $\vec{v}$, $\vec{w}$, $\vec{z}$ all in R^n and let λ and k be scalars.

- Commutative property for vector addition $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
- Associative property for vector addition $(\vec{v} + \vec{w}) + \vec{z} = \vec{v} + (\vec{w} + \vec{z})$
- Existence of additive identity: $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$
- Existence of additive inverse: $\vec{v} + -\vec{v} = -\vec{v} + \vec{v} = \vec{0}$
- Scalar multiplication is commutative: $\lambda (k\,\vec{v}) = k\,(\lambda\,\vec{v})$
- Distributive properties: $\vec{v}\,(\lambda + k) = (\lambda\vec{v} + k\vec{v})$ and $\lambda(\vec{v} + \vec{w}) = (\lambda\vec{v} + \lambda\vec{w})$
- $1 * \vec{v} = \vec{v}$ and $0 * \vec{w} = 0$

Unit vector: any vector with magnitude 1 is a unit vector. Given any vector $\vec{w}$, then the unit vector in the same direction as $\vec{w}$ is given by $\vec{v} = \frac{\vec{w}}{\|\vec{w}\|}$.

Standard unit (basis) vectors: the standard unit vectors for R^n are denoted $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}$ such that all components of every e_k is a zero except the k^{th} entry which is a one. In R^2 $\langle 0,1\rangle$ or $\langle 1,0\rangle$ and in R^3 **i** = $\langle 1,0,0\rangle$ or **j** = $\langle 0,1,0\rangle$ or **k** = $\langle 0,0,1\rangle$ etc.

In R^3 i, j, and k help us with another notation for vectors. If $\vec{v} = \langle a_1, a_2, a_3\rangle$ then we can write

$$\vec{v} = \langle a_1, a_2, a_3\rangle = \langle a_1, 0,0\rangle + \langle 0, a_2, 0\rangle + \langle 0,0, a_3\rangle = a_1\langle 1,0,0\rangle + a_2\langle 0,1,0\rangle + a_3\langle 0,0,1\rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

Ex: if $\vec{a}$ = 2j and $\vec{b}$ = 4i+7k, express the vector $2\vec{a} + 3\vec{b}$ in terms of i, j, and k.

The Dot Product of Two Vectors

The Dot Product:

Let $\vec{v}$ and $\vec{w}$ be vectors in R^n, then the dot product of $\vec{v}$ and $\vec{w}$ is the constant

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

Note: the dot product is not another vector it's a constant and the dot product is also known as the "inner product".

Properties of the Dot Product:

Let $\vec{v}$, $\vec{w}$, and $\vec{z}$ be vectors in R^n and λ be scalar

- $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (commutative)
- $\vec{v} \cdot (\vec{w} + \vec{z}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{z}$ (distributive)
- $\lambda(\vec{v} \cdot \vec{w}) = (\lambda\vec{v}) \cdot \vec{w} = \vec{v} \cdot (\lambda\vec{w})$
- $\vec{0} \cdot \vec{w} = \vec{w} \cdot \vec{0} = 0$
- $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

Ex: Given $\vec{v} = \langle 1, -3\rangle$, $\vec{w} = \langle -2,5\rangle$, $\vec{z} = \langle 1, -3,5,7,9\rangle$, and $\vec{b} = \langle 2, -1, -2, -3,6\rangle$ find $\vec{v} \cdot \vec{w}$, $\vec{w} \cdot \vec{v}$, $\vec{z} \cdot \vec{b}$, $\vec{z} \cdot \vec{v}$, and $\vec{v} \cdot \vec{v}$

Angle Between Vectors:

If Θ is an angle between two nonzero vectors $\vec{v}$ and $\vec{w}$ then $\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}$

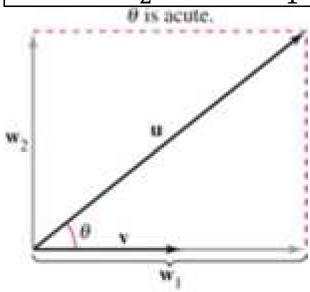
Two vectors are orthogonal (perpendicular) iff $\vec{v} \cdot \vec{w} = 0$ (remember $\cos\pi/2 = 0$)

Ex: Find the angle between $\langle 3, -1,2\rangle$ and $\langle 1, -1, -2\rangle$

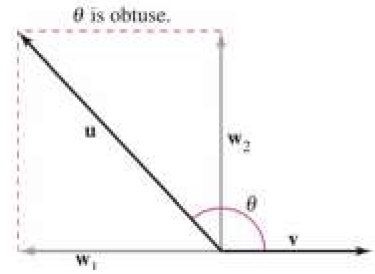
Vector Projections:

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors. Moreover let $\vec{u} = \overrightarrow{w_1} + \overrightarrow{w_2}$, where $\overrightarrow{w_1}$ is parallel to $\vec{v}$, and $\overrightarrow{w_2}$ is orthogonal to $\vec{v}$

1. $\overrightarrow{w_1}$ is called the **projection of u onto v** or the **vector component** of **u** along **v**, and is denoted by $\overrightarrow{w_1}$ = $proj_v u$
2. $\overrightarrow{w_2} = \vec{u} - \overrightarrow{w_1}$ is called the **vector component of u orthogonal to v.**



To project a vector **u** onto a vector **v** we drop a perpendicular from the terminal side of **u** onto the vector **w₁** containing vector **v** then the vector with the same initial point as **u** and **v** to the point of intersection of **v** and **w₁** is the projection of **u** onto **v**



(from here on out vectors will be notated as bolded lower case letter instead of $\vec{v}$)

Formula to find the Vector Projection:

Let **v** and **w** be vectors in R^n that have the same initial point.

$$proj_w v = \left(\frac{v \cdot w}{\|w\|^2}\right) w$$

Ex. Given **v** = [3, -2, 1] and **w** = [4, 1, -5] find $proj_v$**w** and $proj_w$**v**

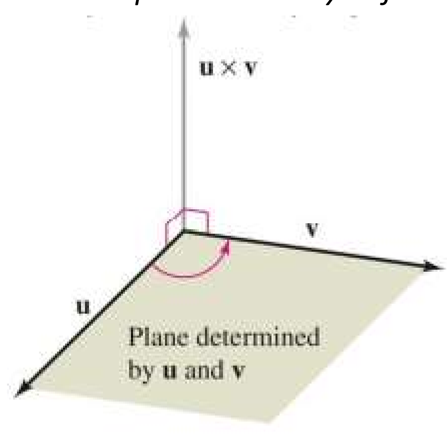
The Cross Product of Two Vectors in Space

The Cross Product

Let **v** and **w** be vectors in R^3, then the cross product of **v** and **w** is

$$\mathbf{v} \times \mathbf{w} = [v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1]$$

Note: the cross product is another vector, the best way to remember the cross product is through determinants, and the cross product is only defined when vectors are in 3-D.



Ex: If **u** = [1, -2, 1] and **v** = [3, 1, -2] find **u** × **v**, **v** × **u**, and **v** × **v**.

<u>Algebraic Properties of the Cross Product:</u>

Let **u**, **v** and **w** be vectors in space, and let c be a scalar

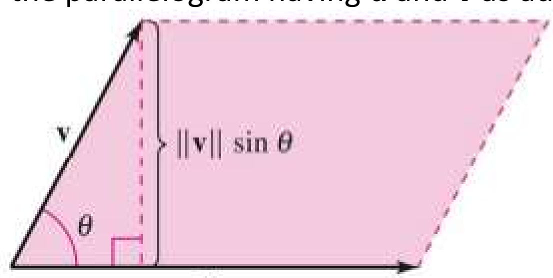
- **u** × **v** = -(**v** × **u**)
- **u** × (**v** + **w**) = (**u** × **v**) + (**u** × **v**)
- c(**u** × **v**) = (c**u**) × **v** = **u** × (c**v**)
- **u** × 0 = 0 × **u** = 0
- **u** × **u** = 0
- **u** · (**v** × **w**) = (**u** × **v**) · **w** *(triple scalar product)*

Since the cross product is another vector we need to find the magnitude and direction of the cross product.

<u>Geometric Properties of the Cross Product</u>

Let **u** and **v** be nonzero vectors in space and let θ be the angle between **u** and **v**

- **u** x **v** is orthogonal to both **u** and **v** *(right hand rule)*
- $\|u \times v\| = \|u\|\|v\| \sin\theta$
- **u** x **v** = 0 iff **u** and **v** are scalar multiples of each other
- **u** x **v** = 0 iff **u** and **v** are parallel
- $\|u \times v\|$ = area of the parallelogram having **u** and **v** as adjacent sides



<u>Triple Scalar Product:</u>

For vectors **u**, **v**, and **w** the triple scalar product is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

(Its easier to solve for this using determinants)

Geometrically the triple scalar product is the volume of a parallelepiped that u, v, and w form in space as long as they all lie on different planes

If the triple scalar product = 0 then two or more of the vectors are on the same plane (coplanar).

Ex: Find the volume of the parallelepiped having **u** = [3, -5, 1] **v** = [0, 2, -2] and **w** = [3, 1, 1] as adjacent edges.

Lines and Planes in Space:

Lines in Space: To find the equation of a line in 2-D we need a point on the line and the slope of the line. Similarly to find the equation of a line in space we need a point on the line and the direction of the line for this we can use a vector.

Given a point P(x_1, y_1, z_1) on a line and a vector **v** = [a,b,c] parallel to the line. The vector **v** is the direction vector for the line and a,b, and c are direction numbers. We can say the line consists of all points Q(x,y,z) for which $\overrightarrow{PQ}$ in parallel to v (a scalar multiple of v), therefore $\overrightarrow{PQ} = t\mathbf{v}$ Where t is a real number.

$$\overrightarrow{PQ} = \langle x - x_1, y - y_1, z - z_1 \rangle = \langle at, bt, ct \rangle = t\mathbf{v}$$

Parametric Equations of a Line in Space

A line L parallel to the vector v = [a,b,c] and passing through the point P[x_1, y_1, z_1] is represented by the parametric equations

$$x = x_1 + at \qquad y = y_1 + bt \qquad z = z_1 + ct$$

Or with the vector parameterization

$$\mathbf{r}(t) = \langle x_1, y_1, z_1 \rangle + t\langle a, b, c \rangle$$

If we eliminate the parameter t we can obtain the symmetric equations

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Ex: Find the parametric and symmetric equations of the line L that passes though the point (2, 1, -3) and is parallel to **v** = [-1, -5, 4]

Ex: Find a set of parametric equations of the line that passes through the points (-3, 1, -2) and (5, 4, -6).

Planes in Space: Like a line in space a plane in space can be obtained from a point in the plane and a vector normal (perpendicular) to the plane.

If a plane contains point P(x_1, y_1, z_1) having a nonzero normal vector n = [a,b,c], this plane consists of all point Q(x,y,z) for which $\overrightarrow{PQ}$ is orthogonal to n. This can be found with a dot product.

$$n \bullet \overline{PQ} = 0$$

$$[a,b,c] \bullet [x - x_1, y - y_1, z - z_1] = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Standard Equation of a Plane in Space

The plane containing the point (x_1, y_1, z_1) and having normal vector **n** = [a, b, c,] can be represented by the **standard form** of the equation of the plane

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Since x_1, y_1, and z_1, are constants the equation can be rewritten as $ax + by + cz + d = 0$ this is known as the **general form**.

Ex: Find the equation of the plane through point (1, 2, 3) and with normal vector
n = [-2, 5, -6]
Ex: Find the equation of the plane that passes through (-1, 2, 4), (1, 3, -1), and (2, -3, 1).

Facts:

- Two planes are parallel iff there normal vectors are scalar vectors of each other
- Two planes intersect in a line
- The angle between two planes is $\cos\theta = \frac{\overrightarrow{n_1}\cdot\overrightarrow{n_2}}{\|\overrightarrow{n_1}\|\|\overrightarrow{n_2}\|}$
- Two planes are orthogonal iff $n_1 \cdot n_2 = 0$

Ex: Find the angle between the two planes

$$x - 2y + z = 0$$
$$2x + 3y - 2z = 0$$

and find the parametric equations of the line of intersection.

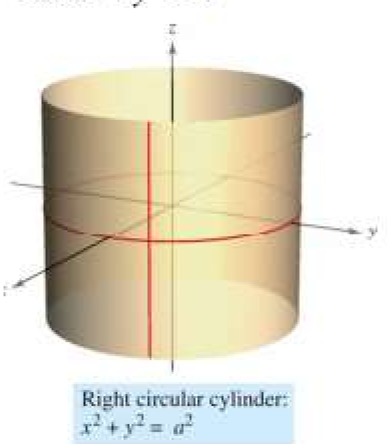
<u>Surfaces in Space:</u>
In previous sections we talked about spheres and planes a third type of surface in space is called a cylindrical surface.

Cylindrical Surface: Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and interesting C is called a **cylinder**. C is called the **generating curve (*directrix*)** of the cylinder, and parallel lines are called **rulings.**

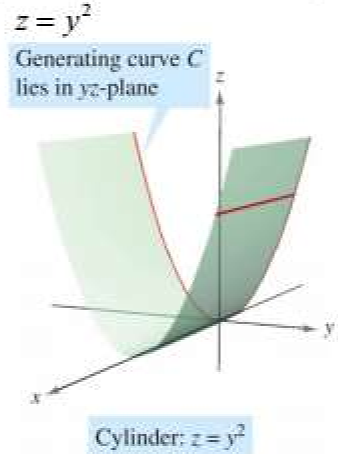
In geometry a cylinder refers to a right circular cylinder but the above definition refer to many types of cylinders.

Equations of Cylinders: The equation of a cylinder whose rulings are parallel to one of the coordinate axes contains only the variables corresponding to the other two axes.

Ex: $x^2 + y^2 = 4$



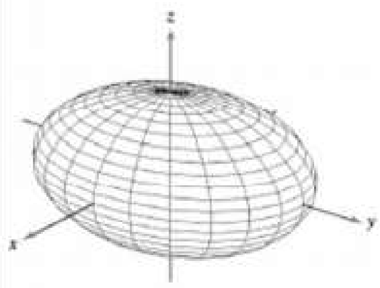
$z = y^2$



Quadric Surface: quadric surfaces are the 3-D counterparts of the conic sections in the plane. The equation of a quadric surface is a second degree equation in three variables. The general form of the equation of a quadric surface is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

The **trace of a surface** is the intersection of the surface with a plane.

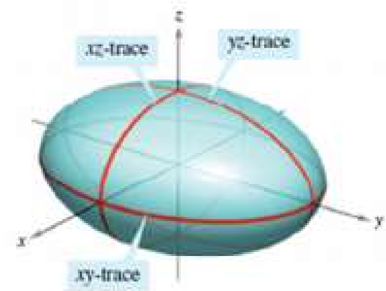


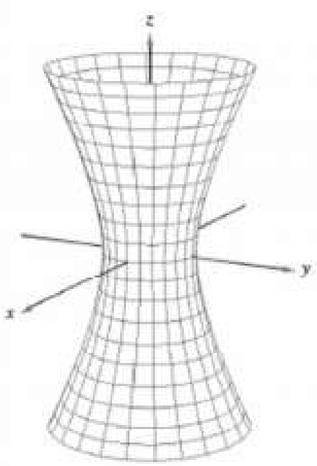
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Ellipse	Parallel to xz-plane
Ellipse	Parallel to yz-plane

The surface is a sphere if $a = b = c \neq 0$.



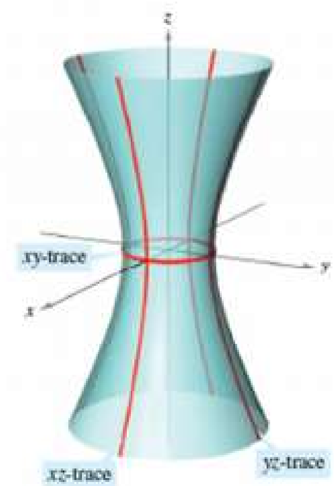


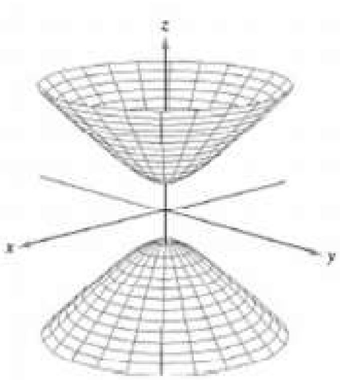
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



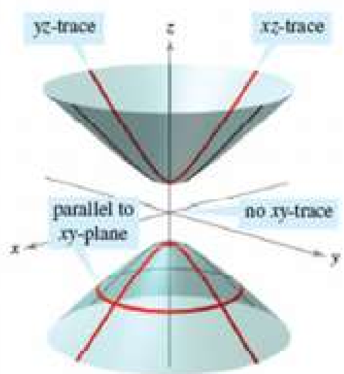


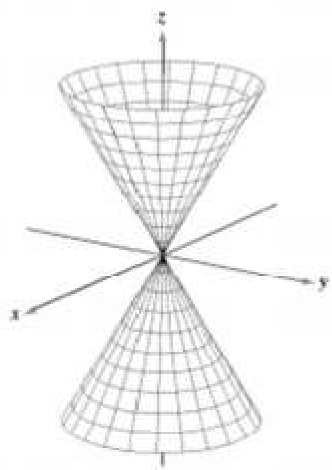
Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	*Plane*
Ellipse	Parallel to xy-plane
Hyperbola	Parallel to xz-plane
Hyperbola	Parallel to yz-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



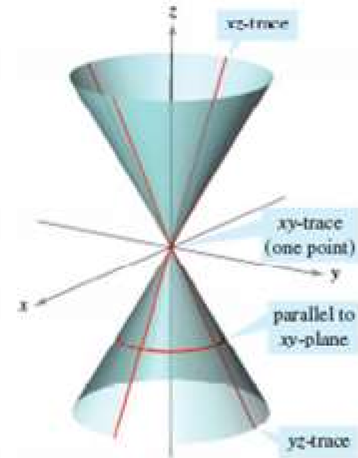


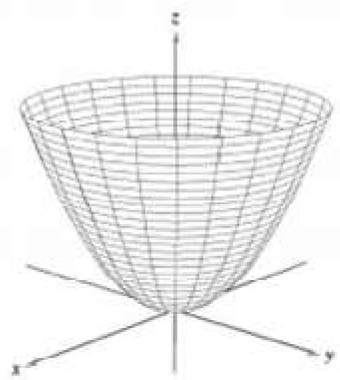
Elliptic Cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

Trace	*Plane*
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



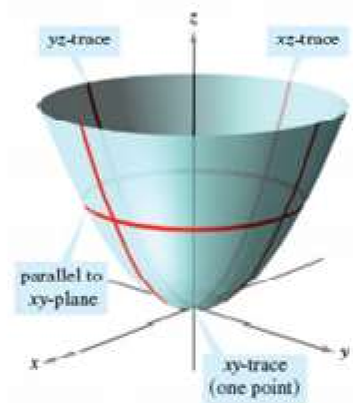


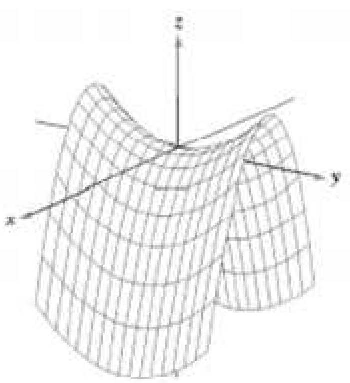
Elliptic Paraboloid

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

Trace	*Plane*
Ellipse	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



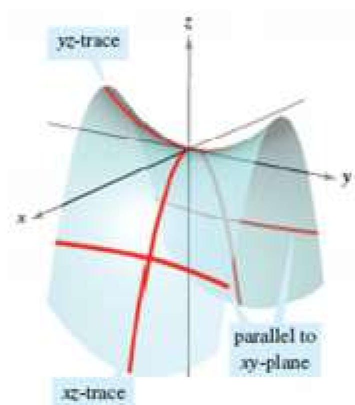


Hyperbolic Paraboloid

$$z=\frac{y^2}{b^2}-\frac{x^2}{a^2}$$

Trace	*Plane*
Hyperbola	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



Ex: Classify the surface

a. $4x^2-3y^2+12z^2+12=0$

b. $16x^2-y^2+16z^2=4$

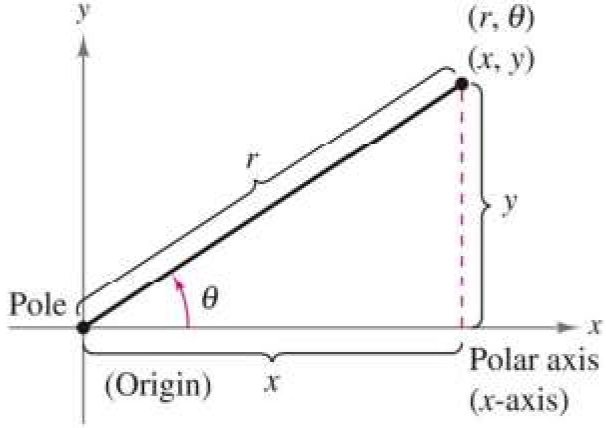
c. $3z+y^2=x^2$

d. $9x^2+y^2-9z^2-54x-4y-54z+4=0$

Cylindrical and Spherical Coordinates:

Review: Polar Coordinates

Polar coordinates (r, θ) of a point (x, y) in the Cartesian plane are another way to plot a graph. The r represents the distance you move away from the origin and θ represents an angle in standard position.



Ex: Convert (3, -3) to polar coordinates

Ex: Convert $\left(2, \frac{2\pi}{3}\right)$ to rectangular coordinates

To convert from polar to rectangular

$x = r\cos\theta$

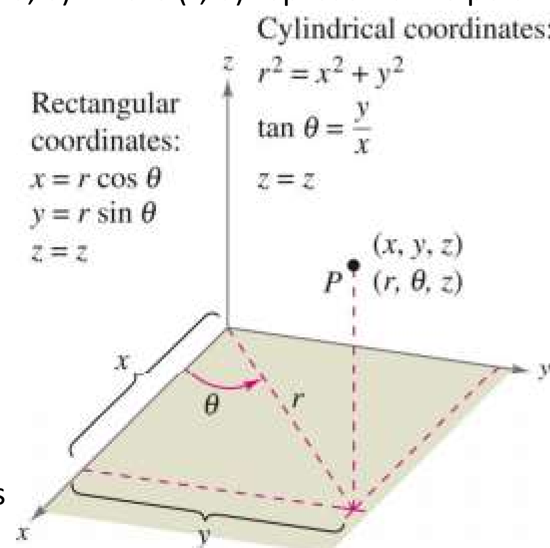
$y = r\sin\theta$

To convert from rectangular to polar

$r^2 = x^2 + y^2$

$\Theta = \arctan(y/x)$

Cylindrical Coordinates: the cylindrical coordinates (r, θ, z) for a point (x, y, z) in the rectangular coordinate system are the point (r, θ, 0) where (r, θ) represents the polar coordinates of (x,y) and z stays the same.



Ex: Convert from cylindrical to rectangular

(2, π/3, -8)

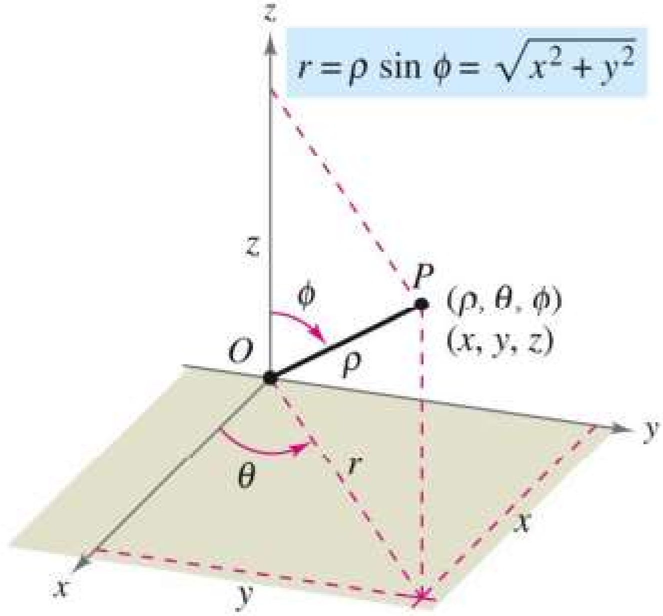
Ex: Find an equation in cylindrical coordinates representing the equation given in rectangular coordinates:

a. $x^2 + y^2 = 9z^2$
b. $y^2 = x$

Ex: Find an equation in rectangular coordinates representing the equation given in cylindrical coordinates $r = \frac{z}{2}$

Spherical Coordinates: the spherical coordinates (ρ, θ, ϕ) for a point (x, y, z) in the rectangular coordinate system is the point (ρ, θ, ϕ) where

- ρ represents the distance from the origin to the point in space
- θ represents the angle in the plane from the positive x – axis to the point (x, y, 0)
- • ϕ represents the angle from positive z-axis to the line segment connecting the origin to the point (x, y, z), $0 \le \phi \le \pi$



To convert from spherical to rectangular

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

To convert from rectangular to spherical:

$\rho = \sqrt{x^2 + y^2 + z^2}$
$\tan\theta = y/x$
$\cos\phi = z/\rho$

To convert from spherical to cylindrical ($r \ge 0$)

$r^2 = \rho^2 sin^2\theta$
$\theta = \theta$
$z = p \cos \emptyset$

Ex: Find the equation in spherical for the equation in rectangular coordinates: $x^2 + y^2 = z^2$