Vector-Valued Functions

Space curves and Vector-Valued Functions:

Recall: a **plane curve** was defined as the set of ordered pairs (f(t), g(t)) together with their defining parametric equations

x = f(t) and y = g(t)

where f and g are continuous functions of t on some interval.

A **space curve** C is the set of all ordered triples (f(t), g(t), h(t)) together with their defining parametric equations

x = f(t)   y = g(t)   z = h(t)

where f, g, and h are continuous functions of t on some interval.

Vector-Valued Function:

A function of the form

$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\,\mathbf{j}$ (plane)

or

$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ (space)

is a **vector-valued function** or vector function, the **component functions** f, g, and h are real valued functions of the parameter t. Vector-valued functions are sometimes denoted $\mathbf{r}(t) = \langle f(t), g(t) \rangle$ or $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

A vector function can be thought of as a function whose domain is the set of real numbers and whose range is a set of vectors.

- Vector-valued functions are similar to real valued functions. The idea is that every vector function just has functions as its components instead of values
- Vector valued functions serve two roles in representing a curve C:
  1. By letting the parameter t be time we can think of the vector function as representing motion along the curve where the particle has position $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ at time t
  2. We can use vector valued functions to *trace out* graphs so that the terminal point of any $\mathbf{r}(t)$ coincides with the point on the curve.

Ex: State the domain of the

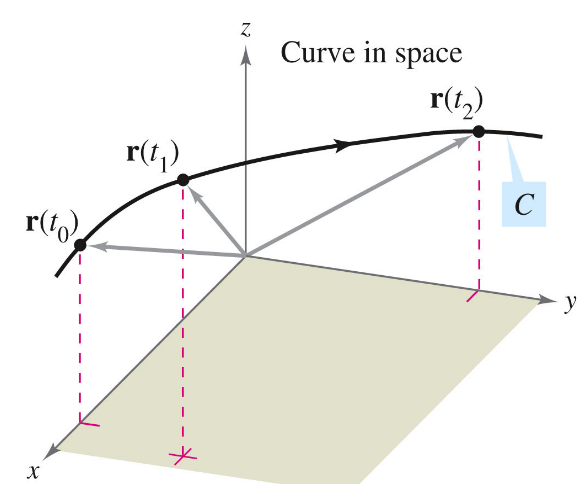
$$\mathbf{r}(t) = \left\langle \ln(1-t), t^3 - 3t + 2, \sqrt{t+4} \right\rangle$$

Ex: Describe the graph and state the domain

a. $\mathbf{r}(t) = [3t - 2, 4 + 5t, 6 - 7t]$

b. $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

c. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$



Limit of Vector-Valued Functions:

If $\boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the components exist.

Ex: Find the $\lim_{t \to 0} \mathbf{r}(t)$ where $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$

Differentiation and Integration of Vector-Valued Functions

The Derivative of a Vector-Valued Function:

If **r**(*t*) is a vector-valued function then

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists

Differentiation of Vector-Valued Functions

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = f(t)'\mathbf{i} + g(t)'\mathbf{j} + h(t)'\mathbf{k}$$

Higher ordered derivatives are defined the same way.

Ex: Find the derivative of $\mathbf{r}(t) = \left(1 + t^3\right)\mathbf{i} + \left(te^{-t}\right)\mathbf{j} + \left(\sin 2t\right)\mathbf{k}$

$\mathbf{r}'(t)$ is called the **tangent vector** to the curve define by **r** at point P, provided $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq 0$. The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$.

Ex: Find the parameterization of the tangent line to $\mathbf{r}(t) = \left(t^3 + t + 2\right)\mathbf{i} + \left(te^{-t}\right)\mathbf{j} + \left(2t - 3\right)\mathbf{k}$ at t = 0

The parameterization of the curve represented by the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is **smooth on an open interval** I if $f', g', and\ h'$ are continuous on I and $\mathbf{r}'(t) \neq 0$ for any value of t in the interval I

Properties of the Derivative:

Let **r**(t) and **v**(t) be differentiable vector-valued functions of t. and let w(t) be a differentiable real valued function of t and let λ be a scalar

- $\frac{d}{dt}\left(\lambda \mathbf{r}(t)\right) = \lambda \frac{d}{dt}\mathbf{r}(t)$
- $\frac{d}{dt}\left(\mathbf{r}(t) \pm \mathbf{v}(t)\right) = \frac{d}{dt}\mathbf{r}(t) \pm \frac{d}{dt}\mathbf{v}(t)$
- $\frac{d}{dt}\left(w(t)\mathbf{r}(t)\right) = w(t)\frac{d}{dt}\mathbf{r}(t) + \mathbf{r}(t)\frac{d}{dt}w(t)$ *w(t) is a real valued function*
- $\frac{d}{dt}\left(\mathbf{r}(t) \bullet \mathbf{v}(t)\right) = \mathbf{r}(t) \bullet \frac{d}{dt}\mathbf{v}(t) + \mathbf{v}(t) \bullet \frac{d}{dt}\mathbf{r}(t)$
- $\frac{d}{dt}\left(\mathbf{r}(t) \times \mathbf{v}(t)\right) = \mathbf{r}(t) \times \frac{d}{dt}\mathbf{v}(t) + \frac{d}{dt}\mathbf{r}(t) \times \mathbf{v}(t)$ *order is very important*
- $\frac{d}{dt}\left(\mathbf{r}(w(t))\right) = \mathbf{r}'(w(t))w'(t)$ *(chain rule)* *w(t) is a real valued function*

Ex: For $r(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$ and $v(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, find $\frac{d}{dt}\left(\mathbf{r}(t) \bullet \mathbf{v}(t)\right)$ and $\frac{d}{dt}\left(\mathbf{r}(t) \times \mathbf{v}(t)\right)$

Integration of Vector-Valued Functions

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ where f, g, and h are continuous on [a,b], then

$$\int \mathbf{r}(t)dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j} + \left[\int h(t)dt\right]\mathbf{k}$$

and

$$\int_a^b \mathbf{r}(t)dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j} + \left[\int_a^b h(t)dt\right]\mathbf{k}$$

Ex: Evaluate

a. $\int \left(2\cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}\right)dt$

b. $\int_0^1 \left(\sqrt[3]{t}\mathbf{i} + \frac{1}{t+2}\mathbf{j} + e^t\mathbf{k}\right)dt$

Velocity and Acceleration

We can now think of a space curve C represented by the vector valued function r(t) as tracing out the motion of a particle as time t increases. Then r(t) is the **position vector** for a particle at time t. Therefore:

velocity vector = $\mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$

acceleration vector = $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$

speed = $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$

Ex: Find the velocity vector, acceleration vector, and speed of

a. $\mathbf{r}(t) = \langle 4t, 3\sin t, -3\cos t \rangle$

b. $\mathbf{r}(t) = \left\langle \ln t, \frac{1}{t}, t^4 \right\rangle$

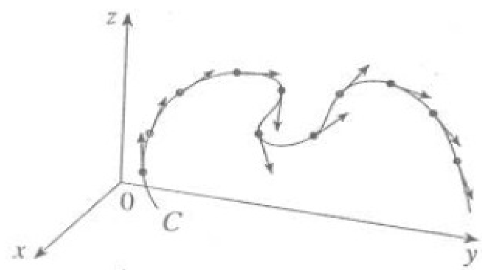
Ex: A moving particle starts at an initial position **r**(0)=[1,0,0] with initial velocity **v**(0) = **i** – **j** +**k**. Its acceleration is **a**(t) = 4t**i** + 6t**j** +**k**. Find its velocity and position at time t.

For an object traveling at a constant speed, the velocity and acceleration vectors are orthogonal to each other $\left(\mathbf{r}'(t)\bullet\mathbf{r}''(t)=0\right)$

Tangent Vectors and Normal Vectors

The Unit Tangent Vector **T**(t): Let C be a smooth curve representing **r**(t) on the open interval (a,b). Then the **unit tangent vector T**(t) is defined by:

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|},\ \mathbf{r}'(t)\neq 0$$



Ex: Find **T**(t) if $\mathbf{r}(t)=\langle 3\sin t, 3\cos t, 4t\rangle$

The (Principal) Unit Normal Vector: Let C be a smooth curve represents by r(t) on the open interval (a,b). Then the **principal unit normal vector N**(t) is defined by:

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{\left\|\mathbf{T}'(t)\right\|}$$

Ex: Find **N**(t) if $\mathbf{r}(t)=\langle 3\sin t, 3\cos t, 4t\rangle$

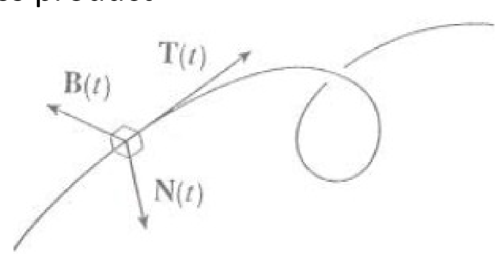
- **N**(t) always points in the direction that to curve is turning (points toward concavity)
- **T**(t) points in the direction the object is moving

The unit tangent vector and the unit normal vector are related in many ways. For example if we think of the unit tangent vector as tangent to the position vector, the unit normal is tangent to the unit tangent vector. While the unit tangent vector is very similar to the velocity vector and points in the same direction, just with magnitude 1. The unit normal vector is only a component of the acceleration vector and doesn't necessarily point in the same direction. The unit tangent and unit normal vectors are orthogonal.

The Binormal Vector: Let C be a smooth curve represented by **r**(t) on the open interval (a,b). Then the binormal vector **B**(t) is defined by the cross product

$$\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)$$

Ex: Find **B**(t) if $\mathbf{r}(t)=\langle 3\sin t, 3\cos t, 4t\rangle$



- **B**(t), **T**(t) and **N**(t) are orthogonal to one another
- The plane determined by **N**(t) and **B**(t) at point P on a curve is called the **normal plane** of C at P. It consists of all the lines that are orthogonal to **T**(t)
- The plane determined by **T**(t) and **N**(t) is called the **osculating plane** of C at P. It is the plane that comes closest to containing the curve C at P.

Remember, for an object traveling at a constant speed, the velocity and acceleration vectors are orthogonal to each other $(\mathbf{r}'(t) \bullet \mathbf{r}''(t) = 0)$. However, for an object traveling at a variable speed this is not necessarily the case.

The velocity vector can change in two ways: in magnitude and direction. These changes are "encoded in the acceleration vector.

- part of the acceleration vector acts in the line of motion, the tangential component = a_T is the rate at which the speed (magnitude) changes
- part of the acceleration vector acts orthogonal to the line of motion, the normal component = a_N describes the change in direction.

If **r**(t) is the position vector for a smooth curve C and **N**(t) exists, then the acceleration vector **a**(t) lies in the plane determined by **T**(t) and **N**(t) *(the osculating plane).*

Let $\boldsymbol{\upsilon}(t) = \|\vec{v}(t)\| = speed$ then $v(t) = \boldsymbol{\upsilon}\vec{\mathbf{T}}$ so

$$\begin{aligned} \vec{a} = \vec{v}' &= \boldsymbol{\upsilon}'\vec{\mathbf{T}} + \boldsymbol{\upsilon}\mathbf{T}' \qquad (product\ \ rule) \\ &= \boldsymbol{\upsilon}'\vec{\mathbf{T}} + \boldsymbol{\upsilon}\|\mathbf{T}'\|\vec{\mathbf{N}} \\ &= a_{\mathrm{T}}\vec{\mathbf{T}} + a_{\mathrm{N}}\vec{\mathbf{N}} \end{aligned}$$

This shows that $\vec{a}$ lies in the plane of **T** and **N:** $a_T = \mathrm{v}' = \frac{d}{dt}\|\vec{v}(t)\|$ and easier calculation of a_N is $a_N = \|\vec{v} \times \vec{a}\| / \|\vec{v}\|$

Ex: Decompose the acceleration vector of $\mathbf{r}(t) = \left\langle t^2, 2t, \ln t \right\rangle$ into the tangential and normal components at t = ½

Arc Length and Curvature:

Arc Length of a Space Curve:

If C is a smooth curve given by , on an interval [a,b], then the arc length on the interval is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b \|r'(t)\|\, dt$$

Ex: Find the length of the curve $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle$ on [0, π]

Ex: Find the arc length of $\mathbf{r}(t) = \left\langle \cos t + t\sin t, \sin t - t\cos t, t^2 \right\rangle$ over [0, π/2]

Arc Length Function:

Let C be a smooth curve given by **r**(t) defined on the closed interval [a,b]. For a ≤ t ≤ b, the arc length function is given by

$$s(t) = \int_a^t \|r'(u)\| du = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du$$

You can see that $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$

Ex: Find the arc length function for $\mathbf{r}(t) = (3 - 3t)\mathbf{i} + 4t\mathbf{j}$

<u>Curvature</u>:

The **curvature** of C at a given point is a measure of how quickly the curve changes direction or how sharply the curve bends at that point. You can find the curvature by the change in **T**(t) with respect to the arc length and is found using

$$K = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

- the curvature is always a real number (scalar) at a given value of t
- for a circle the curvature is always the same at any given point and is given by 1/r

Ex: Find the curvature for $\mathbf{r}(t) = \langle 3\sin t, 3\cos t, 4t \rangle$