# Vector Analysis

## Vector Fields

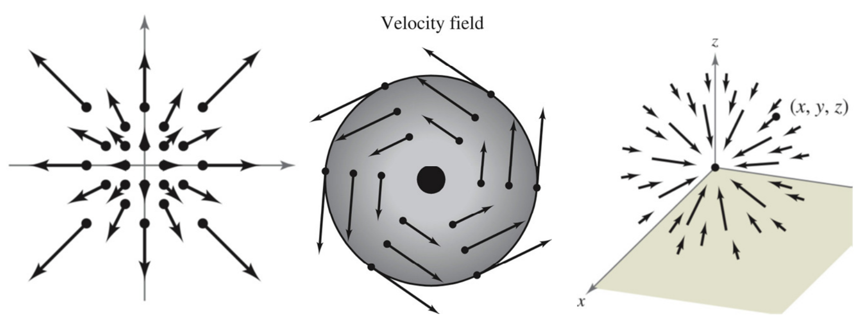
Suppose a region in the plane or space is occupied by a moving "fluid" such as air or water. Imagine this "fluid" is made up of a very large number of particles that at any instant of time a given particle has velocity vector **v.** The set of these velocity vectors is what we call a **vector field.** If we examine these velocities we understand that they will vary from position to position.

Some common examples of vector fields: wind shear off an object, gravitational fields, electric and magnetic fields, etc…

**Vector Field:**
A **vector field** in $R^2$ or $R^3$ respectively is a function **F** that assigns to each point $(x,y)$ or $(x,y,z)$ respectively a 2-dimensional or 3-dimensional vector $\mathbf{F}(x,y)$ or $\mathbf{F}(x,y,z)$ where

$$\mathbf{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle \text{ or } \mathbf{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$$

In general, a vector field is a function whose domain is the set of points in $R^2$ or in $R^3$ and whose range is a set of vectors in $V^2$ or $V^3$.




A **unit vector field** is a vector field **F** such that $\|\boldsymbol{F}(P)\|=1$ for all points P in the domain.
A vector field **F** is called a **radial vector field** if **F**(P) depends only on a distance r from point P to the origin.
An important example of a unit radial vector field is:

$$e_r = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\rangle \text{ where } r = \sqrt{x^2+y^2+z^2}$$

We have already worked with one type of vector field in the gradient. The **gradient vector field** of a differentiable function $f(x, y, z)$ is the field of gradient vectors given by:

$$\nabla f = \boldsymbol{F}(x,y,z) = \langle f_x, f_y, f_z\rangle$$

This type of vector field is also known as a **conservative vector field**, and $f$ is called a **potential function** for $\nabla f$.

**Ex:** Find the gradient vector field for the potential function

$f(x, y, z) = xy + yz^2$

How would we know a vector field is conservative, that is it came from the partial derivatives of a potential function?

<u>**Cross-Partial Property of Conservative Vector Fields:**</u>

If a vector field $\mathbf{F}(x, y, z) = \langle F_1, F_2, F_3\rangle$ is conservative, then

$$\frac{\partial \boldsymbol{F}_1}{\partial y} = \frac{\partial \boldsymbol{F}_2}{\partial x}, \frac{\partial \boldsymbol{F}_2}{\partial z} = \frac{\partial \boldsymbol{F}_3}{\partial y}, \frac{\partial \boldsymbol{F}_3}{\partial x} = \frac{\partial \boldsymbol{F}_1}{\partial z}$$

If $\mathbf{F}(x, y) = \langle F_1, F_2\rangle$ then only the first equality has to hold

**Ex:** Show that $\mathbf{F}(x, y, z) = \langle 3, 1, 2\rangle$ is conservative. Would any constant vector field be conservative?

**Ex:** Is $\mathbf{F}(x, y) = \langle 5y^3, 15xy\rangle$ conservative?

**Ex:** Is $\mathbf{F}(x, y, z) = \langle e^x \cos y, -e^x \sin y, 2\rangle$ conservative?

Let **F** be a vector field on a simply connected domain D (domain has no "holes"). If **F** satisfies the cross partials condition, then **F** is conservative and therefore has a potential function.

If $f$ is a potential function for **F** that $\frac{\partial f}{\partial x} = \boldsymbol{F}_1, \frac{\partial f}{\partial y} = \boldsymbol{F}_2$, etc. or another way to look at it is $\int F_1 dx = \int F_2 dy = \ldots.. = f(x, y, \ldots)$

**Ex:** Determine if $\mathbf{F}(x, y) = \langle xy^2, x^2y\rangle$ is conservative, if so find its potential function.
**Ex:** Determine if $\mathbf{F}(x, y, z) = \langle yz, xy, xy + 2z\rangle$ is conservative, if so find its potential function.

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<u>Line Integrals</u>

A **line integral** (curve integral) is similar to a single integral except instead of integrating over an interval we integrate over a curve. These integrals are used to solve problems like fluid flow, electricity, forces and magnetism.
Like all integrals this line integral is defined through a process of subdivisions, summations, and limits. We divide $C$ into $n$ consecutive arcs, choose a sample point $P_i$ in each arc $C_i$ and form a Riemann sum.

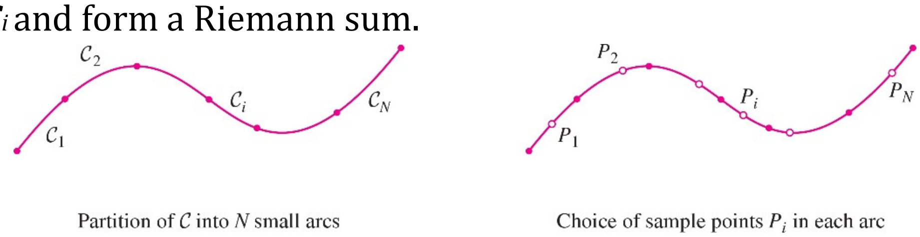


**FIGURE 1** The curve $C$ is divided into $N$ small arcs.

$$\sum_{i=1}^{n} f(P_i)(length\ of\ C_i) = \sum_{i=1}^{n} f(P_i)\Delta s_i$$

The line integral of $f$ over $C$ is

$\int_C f(x,y,z)ds = \lim_{\Delta s_i \to 0} \sum_{i=1}^{n} f(P_i)\Delta s$ where $ds$ is the arc length differential

**Computing a Scalar Line Integral:**

Let $\mathbf{r}(t)$ be a parameterization of a curve $C$ for $a \le t \le b$. If $f(x,y,z)$ and $\mathbf{r}'(t)$ are continuous, then

$$\int_C f(x,y,z)ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|dt$$

**Ex:** Evaluate $\int_C \sqrt{x^2+y^2}\,ds$ along the curve $(t) = \langle 4\cos t, 4\sin t, 3t\rangle$ ; $-2\pi \le t \le 2\pi$

**Ex:** Evaluate $\int_c (xy+y+z)ds$ along $\mathbf{r}(t) = \langle 2t, t, 2-2t\rangle$ on $0 \le t \le 1$

## Vector Line Integrals

When you carry a backpack up a mountain you do work against the earth's gravitational field. The work or energy expended is an example of quantity represented by a vector line integral (work done by force).

**Computing a Vector Line Integral (Work Done by Force over a Curve in Space):**

Suppose $\mathbf{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$ represents a force throughout a region in space and suppose $\mathbf{r}(t) = \langle g(t), h(t), k(t)\rangle$, $a \le t \le b$ is a smooth curve C in the region. Then the work done by **F** over C defined by $\mathbf{r}(t)$ from $a$ to $b$ is:

$$W = \int_C \mathbf{F}ds = \int_C \mathbf{F} \bullet \mathbf{T}ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt$$

This integral is known as a vector line integral.

Another notation is $\int_C \mathbf{F}ds = \int_C F_1dx + F_2dy + F_3dz$

Two big differences between vector line integrals and a line integrals (scalar line integrals) is scalar line integrals are integrating functions and vector line integrals are integrating vector fields over curves and, that a vector line integral depends on a direction along a curve.

**Ex:** Find the vector line integral for $\mathbf{F}(x,y,z) = \langle x,y,z\rangle$ along the path $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le 3\pi$.

**Ex:** Find the work done by **F** from (0,0,0) to (1,1,1) over the curve $\mathbf{r}(t) = \langle t, t^2, t^4\rangle$ where $\mathbf{F} = \langle 3x^2 - 3x, 3z, 1\rangle$ .

## Conservative Vector Fields

When a curve C is **closed**, has the same start and end point, then we say the line integral of **F** is the **circulation** of **F** around C.
Remember a conservative vector field **F** possesses a function $f$ such that **F**=$\nabla f$

**Fundamental Theorem of Conservative Vector Fields:**
Assume the **F**=$\nabla f$ on a domain D, so $f$ is the potential function for **F**.

1. If C is a path from point $P$ to point $Q$ in D then

$$\int_C \mathbf{F} ds = V(Q) - V(P)$$

**F** is path-independent (all that matters is the start and end points)

2. The circulation around a closed path C ($P$=$Q$) is zero:

$$\oint_C \mathbf{F} ds = 0$$

**Ex:** Let $\mathbf{F}(x, y, z) = \langle 2xy + z, x^2, x \rangle$, evaluate $\int_C \mathbf{F} ds$ where $C$ is a path from $P$= (1,-1,2) to $Q$=(2,2,3).
**Ex:** Determine if $\mathbf{F}(x, y, z) = \langle 2xy - z, x^2 + 2y, 1 - x \rangle$ is conservative, if so find its potential function and evaluate its vector line integral where C is a curve from (1, 0, 2) to (2, 1, 3).

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## Green's Theorem

Remember that for a conservative vector field the circulation around a closed path is zero. For vector fields in the plane, Green's Theorem tells us what happens when F is not conservative.

A **simple closed curve** is a curve that does not intersect itself.

If a domain D has boundary $C$ where $C$ is a simple closed curve then we can denote $C$ as $\partial D$. The counterclockwise orientation of D is called the **boundary orientation.**
Notation:
If $\mathbf{F}(x, y) = \langle F_1, F_2 \rangle$ then

$$\int_C \mathbf{F} ds = \int_C F_1\, dx + F_2 dy$$

**Green's Theorem:**
Let D be a domain whose boundary $\partial D$ is a simple closed curve, oriented counterclockwise. Then

$$\oint_{\partial D} \mathbf{F} ds = \oint_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

**Ex:** Compute the circulation of $\mathbf{F}(x, y) = \langle \sin x, x^2\, y^3 \rangle$ around the triangular path $C$, where $C$ goes from point (0, 0) to (2, 0) to (2, 2) to (0, 0).

**Ex:** Use Green's Theorem to evaluate

$$\int_C (\arctan(x^2) - y^2)dx + (x^2y - \log(y^2+1)dy$$

where C is the semicircle $y = \sqrt{4-x^2}$ together with the line segment form (-2, 0) to (2, 0).

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<u>Stokes' Theorem</u>

Stokes' Theorem can be regarded as a higher dimensional version of Green's Theorem. Whereas Green's Theorem relates to double integrals over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates to a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve).

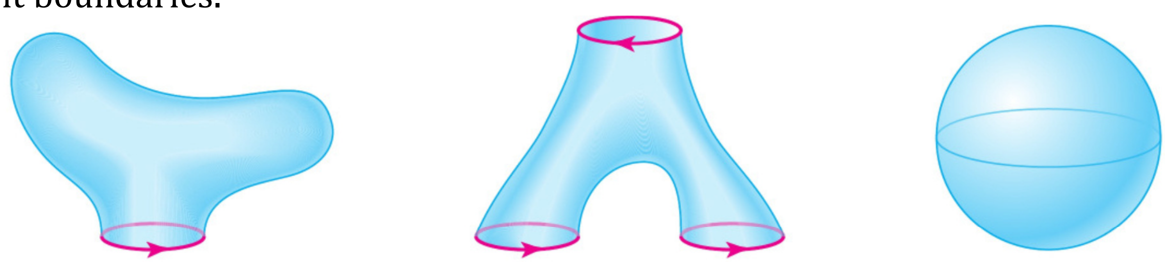
<u>Surface Integrals</u>

Rather than integrating a function or vector field over a curve we can integrate them over a surface. Surface integrals of vector fields represent **flux** or rates of flow through the surface, such as molecules across a cell membrane.
Because flux goes through a surface from one side to another we need to specify orientation of flow in the positive direction. The normal vector $\mathbf{e}_n$ at a point on the surface points in the direction of orientation.

A vector surface integral is defined as $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{F} \cdot \mathbf{e_n}) dS$

To compute this integral you would have to parameterize the surface S.

Stokes' theorem is an extension of Green's theorem to three dimensions in which circulation is related to a surface integrals. The following depicts three surfaces with different boundaries.



(A) Boundary consists of a single closed curve (B) Boundary consists of three closed curves (C) Closed surface (the boundary is empty)

**FIGURE 1** Surfaces and their boundaries.

When S is oriented, we can specify an orientation of $\partial S$ called the boundary orientation. Imagine that you are a unit normal vector walking along a boundary curve. The boundary orientation is the direction for which the surface is on your left as you walk.

The last thing we need to define is the curl. The **curl** of a vector field $\mathbf{F}(x, y, z) = \langle F_1, F_2, F_3 \rangle$ is the vector field defined by the symbolic determinant

$$curl(\mathbf{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \text{ or } curl(\mathbf{F}) = \nabla \times \mathbf{F} \text{ where } \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

**Ex:** Calculate the curl of $\mathbf{F} = \langle xy, e^x, y + z \rangle$

If **F** is conservative then *curl*(F) = 0

**Stokes' Theorem:**

Let S be oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let **F** be a vector field whose components have continuous partial derivatives on an open region in $R^3$ that contains S. Then

$$\oint_{\partial s} \mathbf{F} \cdot ds = \iint_{S} curl(\mathbf{F}) \cdot d\mathbf{S}$$

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## The Divergence Theorem

The divergence theorem is an extension of Stokes' theorem for closed surfaces and triple integrals.

One term we need to define is the divergence of a vector field $\mathbf{F}(x, y, z) = \langle F_1, F_2, F_3 \rangle$ denoted div(**F**).

$$div(\mathbf{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \mathbf{F}$$

**Ex:** Evaluate the divergence of $\mathbf{F} = \langle e^{xy}, xy, z^4 \rangle$

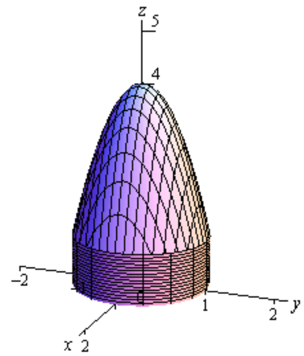
**The Divergence Theorem**

Let S be a closed surface that encloses a region W in $R^3$. Assume S is piecewise smooth and is oriented by normal vectors pointing to the outside of W. Let F be a vector field whose domain contains W. Then

$$\iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iiint_{W} div(\mathbf{F}) dV$$

**Ex:** Find the flux of a vector field $\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

**Ex:** Use the divergence to evaluate $\iint_S \mathbf{F} dS$ where




$\mathbf{F}(x, y, z) = \langle xy, -\frac{1}{2}y^2, z\rangle$ and the surface consists of the three surfaces, $z = 4 - 3x^2 - 3y^2, 1 \le z \le 4$ on the top, $x^2 + y^2 = 1$, $0 \le z \le 1$ on the sides, and $z = 0$ on the bottom.

## Summary of the "Fundamental Theorems"

**The Fundamental Theorem of Calculus:**

$\int_a^b F'(x)dx = F(b) - F(a)$ where [a, b] is an interval over the x axis

**The Fundamental Theorem of Line Integrals:**

$\int_c \nabla f ds = f(Q) - f(P)$ where P and Q are end points over a smooth curve

**Green's Theorem:**

$\oint_{\partial D} \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$ where D is the region bounded by a simple closed curve

**Stokes' Theorem:**

$\oint_{\partial S} \mathbf{F} ds = \iint_s curl(\mathbf{F}) d\mathbf{S}$ where S is the surface with the boundary curve C

**Divergence Theorem:**

$\iint_s \mathbf{F} \cdot d\mathbf{S} = \iiint_W div(\mathbf{F}) dV$ where S is the boundary surface of the 3-D region W