Trigonometry

Trigonometry comes from the Greek word meaning measurement of triangles

Angles are typically labeled with Greek letters

$$\alpha\,(alpha),\quad \beta\,(beta),\ \theta\,(theta)$$

as well as upper case letters A,B, and C

The **measure of an angle** is determined by the **amount of rotation** from the initial side to the terminal side.

Degree Measure

The amount of rotation in an angle with measure one **degree,** denoted by 1^0, is equivalent to the rotation in 1/360 of an entire circle about the vertex.

a **full revolution = 360^0**
a **half of a revolution = 180^0**
a **quarter of a revolution = 90^0**
and so on…..

The most common angles you will see are multiples of 30^0, 45^0, and 60^0

Radian Measure

Another way to measure angles is in **radians**, which is useful in calculus. To define we use a central angle of a circle, one whose vertex is in the center of the circle

Definition:

One **radian** is the measure of a central angle θ that intercepts an arc ***s*** equal in length to the radius ***r*** of a circle.

In general, the radian measure of a central angle θ is obtained be dividing the arc length *s* by *r*, that is $s / r = \theta$ measured in radians.

Since the circumference of a circle is $2\pi r$ the arclength s (of the entire circle) is $2\pi r$. Because $2\pi \approx 6.28$, there are just over six radius lengths in a full circle.

One full revolution has radian measure of 2π

a **half of a revolution** = $\frac{2\pi}{2} = \pi$ radians

a **quarter of a revolution** = $\frac{2\pi}{4} = \frac{\pi}{2}$ radians

a **sixth of a revolution** = $\frac{2\pi}{6} = \frac{\pi}{3}$ radians

Conversions between Degrees and Radians

1. Convert **Degrees to Radians**: multiply degrees by $\frac{\pi}{180^0}$

2. Convert **Radians to Degrees**: multiply radians by $\frac{180^0}{\pi}$

Remember: we do not write the units for radian measure

The Unit Circle:

The unit circle is a circle centered at the origin with radius 1 so the equation of this circle would be $x^2 + y^2 = 1$. We use this circle to help us define the six trigonometric functions.

We start by working with the two most basic trigonometric functions the **sine of θ**, written as**, sin(θ)** or **sinθ** and **cosine of θ** written as **cos(θ)** or **cosθ**, where θ is an angle in standard position.

> If **θ** is an angle in standard position and (x,y) is the point of intersection of the terminal side and the unit circle, then
>
> **sinθ = y** and **cosθ = x**

The domain of both sine and cosine is the set of all angles in standard position or $(-\infty,\infty)$ and the range for each is $[-1,1]$

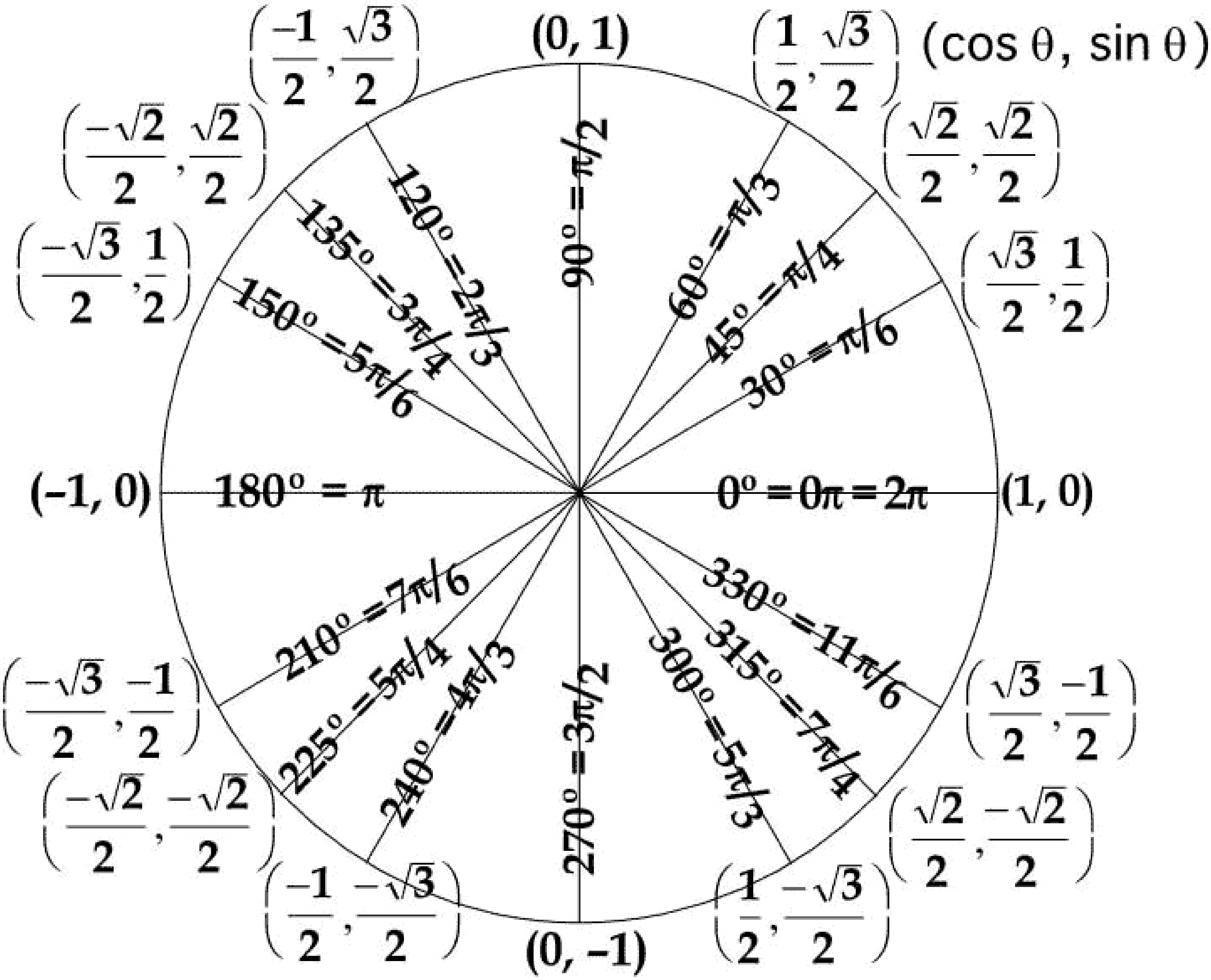
There are four other trigonometric functions. They are tangent (tan), cotangent (cot), secant (sec), and cosecant (csc). These can also be defined using the unit circle.

> If θ is an angle in standard position and (x,y) is the point of intersection of the terminal side and the unit circle, we define the tangent, cotangent, secant, and cosecant as
>
> $$\tan\theta = \frac{y}{x},\ \cot\theta = \frac{x}{y},\ \sec\theta = \frac{1}{x},\ \csc\theta = \frac{1}{y}$$
>
> Since these functions are equal to fractional values we need to restrict the domains to keep the denominators from being zero.

Since sinθ = y and cosθ = x we can rewrite each of the previous in terms of sine and cosine

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \csc\theta = \frac{1}{\sin\theta}$$

again providing that the denominator is not zero.



Θ degrees	**0°**	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**
Θ radians	**0**	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
sin Θ	**0**	**½**	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	**1**	**0**	**– 1**
cos Θ	**1**	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	**½**	**0**	**– 1**	**0**
tan Θ	**0**	$\frac{\sqrt{3}}{3}$	**1**	$\sqrt{3}$	**Undefined**	**0**	**Undefined**

Signs of the trigonometric functions in the four quadrants.

- Both *x* and *y* are positive in the first quadrant, all six functions are positive in the first quadrant.
- Only *y* is positive in the second quadrant, only sine and cosecant are positive in the second quadrant.
- Both *x* and *y* are negative in the third quadrant, only tangent and cotangent are positive in the third quadrant.
- Only *x* is positive in the fourth quadrant, only cosine and secant are positive in the fourth quadrant.

Definitions of Trigonometric Functions of any angle

Let θ be an angle in standard position with (*x*, *y*) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y} \quad y \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x} \quad x \neq 0$$

$$\tan\theta = \frac{y}{x} \quad x \neq 0 \qquad \cot\theta = \frac{x}{y} \quad y \neq 0$$

Ex: Let (4, -3) be on the terminal side of θ. Find the value of the sine, cosine, and tangent of θ.

Ex: Given cos θ = 3/5 and tan θ < 0, find sin θ.

Right Triangles

Hypotenuse (hyp) - side connecting angle theta and other non right angle
Opposite (opp) - side connecting right angle and other non right angle
Adjacent (adj) - side connecting theta and right angle

Pythagorean Theorem:

$$a^2 + b^2 = c^2 \quad or \quad (opp)^2 + (adj)^2 = (hyp)^2$$

The Six Trigonometric Functions and Right Triangles

Let θ be an acute angle of a right triangle, the six trig functions of the angle θ are defined as follows:

$$\mathbf{sin\theta} = \frac{opp}{hyp} \qquad \mathbf{csc\theta} = \frac{hyp}{opp}$$

cosθ = $\frac{adj}{hyp}$ **secθ =** $\frac{hyp}{adj}$

tanθ = $\frac{opp}{adj}$ **cotθ =** $\frac{adj}{opp}$

opp = the length of the side *opposite* θ
adj = the length of the side *adjacent to* θ
hyp = the length of the *side that does not touch the* 90^0 *angle*
Look at right triangles: $45^0,45^0,90^0$ and $30^0,60^0,90^0$
And evaluate the six trig functions for each theta.

Trigonometric Identities

We use trig. identities to:

1. Evaluate trig. functions
2. Simplify trig. expressions
3. Develop additional trig. identities
4. Solve trig. Equations

Reciprocal Identities

sinθ = $\frac{1}{\csc\theta}$ cosθ = $\frac{1}{\sec\theta}$ tanθ = $\frac{1}{\cot\theta}$

cscθ = $\frac{1}{\sin\theta}$ secθ = $\frac{1}{\cos\theta}$ cotθ = $\frac{1}{\tan\theta}$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad , \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \quad , \quad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

The pythagorean identities can also be expressed as radicals

Even/Odd Identities

sin (-*x*) = - sin *x*	tan (-*x*) = - tan *x*	*(odd)*
cot (-*x*) = - cot *x*	csc (-*x*) = - csc *x*	*(odd)*
cos (-*x*) = cos *x*	sec (-*x*) = sec *x*	*(even)*

Cofunction Identities

$$\sin\left(\frac{\pi}{2}-u\right)=\cos u \qquad \cos\left(\frac{\pi}{2}-u\right)=\sin u$$

$$\tan\left(\frac{\pi}{2}-u\right)=\cot u \qquad \cot\left(\frac{\pi}{2}-u\right)=\tan u$$

$$\sec\left(\frac{\pi}{2}-u\right)=\csc u \qquad \csc\left(\frac{\pi}{2}-u\right)=\sec u$$

Ex: If **sec *u* = -5/3** and **tan *u* > 0**, find the values of the other five trigonometric functions.

Ex: Show that this is not an identity

$$\tan^2\theta-1=\sec^2\theta$$

Ex: Verify the identity $\dfrac{\sec^2\theta-1}{\sec^2\theta}=\sin^2\theta$

Ex: Verify the identity $2\sec^2\beta=\dfrac{1}{1-\sin\beta}+\dfrac{1}{1+\sin\beta}$

Solving Trigonometric Equations

A **solution** to any equation is any value that can be plugged in for the variable(s) that make the equation true, basically making one side equal to the other.

Ex: Find all the solutions to

a. $\cos x=1$ **b.** $\cos x=0$ **c.** $\cos x=-1/2$

Solving cosx = *a*

1. If -1 < *a* < 1 and *a* ≠ 0, the solution set is $\{x \mid x=s+2k\pi\}$, where s = $\cos^{-1}a$

2. The solution set to cosx = 1 is $\{x \mid x=2k\pi\}$

3. The solution set to cosx = 0 is $\{x \mid x=\pi/2+k\pi\}$

4. The solution set to cosx = -1 is $\{x \mid x=\pi+2k\pi\}$

5. If $|a|>1$, then cosx = *a* has **NO** solution.

Ex: Find all the solutions to $\sin x=-1/2$

Solving sinx = *a*

1. If -1 < *a* < 1, *a* ≠ 0 and s = sin^{-1}*a* the solution set is $\{x|x = s + 2k\pi\}$ for s > 0 and $\{x|x = \pi - s + 2k\pi\}$ for s < 0.

2. The solution set to sinx = 1 is $\{x|x = \pi/2 + k\pi\}$

3. The solution set to sinx = 0 is $\{x|x = k\pi\}$

4. The solution set to sinx = -1 is $\{x|x = 3\pi/2 + k\pi\}$

5. If $|a| > 1$, then sinx = *a* has **NO** solution.

Solving tanx = *a*

If *a* is any real number and s = tan^{-1}*a*, then the solution set to tanx = *a* is $\{x|x = s + k\pi\}$ for s ≥ 0, and $\{x|x = s + \pi + 2k\pi\}$ for s < 0.

Ex: Find all the solutions to $\sin 2\theta = \frac{\sqrt{2}}{2}$

Ex: Find all the solutions to $\tan 3x = \sqrt{3}$

Ex: Find all the solutions in the interval $[0, 2\pi]$ **to** $\sin 2x = \sin x$

Ex: Find all the solutions to $6\cos^2 x - 7\cos x + 2 = 0$

Ex: Find all the solutions in the interval [0,360^0] that satisfy the equation $\tan 3y + 1 = \sqrt{2}\sec 3y$

Solving Trigonometric equations

1. Know the solutions to sinx = a, cosx = a, tanx = a.
2. Solve an equation involving multiple angles as if it had a single variable.
3. Simplify complicated equations by using identities. Try to get and equations with a single trigonometric function.
4. If possible use factoring and the zero product property.
5. Square each side of the equation if necessary, so you can use identities with squares.