The Natural Logarithmic and Exponential Functions: Differentiation and Integration

Objective: Find derivatives of functions involving the natural logarithmic function.

The Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

$$\frac{d}{dx}[\ln x]=\frac{1}{x}, x>0 \quad , \quad \frac{d}{dx}[\ln u]=\frac{1}{u}\frac{du}{dx}=\frac{u'}{u},\ u>0 \quad , \quad \frac{d}{dx}[\ln|u|]=\frac{u'}{u},\quad u\neq 0$$

Derivative of the Natural Exponential Function

Let u be a differentiable function of x.

$$\frac{d}{dx}\left[e^x\right]=e^x \quad , \quad \frac{d}{dx}\left[e^u\right]=e^u\frac{du}{dx}$$

Ex: Differentiate :

a. $y=\ln(5x^2+2)$

b. $f(x)=\ln\frac{x(2-x^2)^3}{\sqrt{5x^4-3}}$

c. $y=\frac{(x-3)^2}{\sqrt{x^3+2}}, x\neq 3$

d. $y=e^{5x^2+x-1}$

e. $y=\ln(\ln x)$

f. $y=x^3e^{x^2}$

Ln Rule for Integration

Let u be a differentiable function of x

$$\int\frac{1}{x}dx=\ln|x|+C \quad , \quad \int\frac{1}{u}du=\ln|u|+C$$

Integration Rules for Exponential Functions

Let u be a differentiable function of x.

$$\int e^x dx=e^x+C \quad , \quad \int e^u du=e^u+C$$

Ex: Evaluate

a. $\int\frac{4x^3+1}{x^4+x}dx$

b. $\int\frac{\csc^2 x}{\cot x}dx$

c. $\int\frac{x^3+2x^2-x+1}{x^2+1}dx$

Ex: Find the area of the region bounded by the graph $y = \frac{2x}{x^2+3}$, the x-axis, the line $x = 3$.

Ex: Evaluate

a. $\int \frac{e^{1/x}}{x^2}\,dx$ **b.** $\int \cos x e^{\sin x}\,dx$ **c.** $\int \frac{e^{3x} - 3e^{2x} + e^x}{e^{2x}}\,dx$ **d.** $\int_0^1 \frac{e^x}{1+e^x}\,dx$

<u>Integrals of the Six Basic Trig Functions</u>

1. $\int \sin u\,du = -\cos u + C$
2. $\int \cos u\,du = \sin u + C$
3. $\int \tan u\,du = -\ln|\cos u| u + C$
4. $\int \cot u\,du = \ln|\sin u| u + C$
5. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
6. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

Bases Other than e and Applications

Objective: Define exponential functions that have bases other than e. Differentiate and integrate exponential functions that have bases other than e. Use exponential functions to model compound interest and exponential growth.

<u>Definition of Exponential Function to Base a:</u>

If a is a positive real number (a ≠ 1) and x is any real number, then the exponential function to the base a is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}$$

<u>Definition of Logarithmic Function to Base a:</u>

If a is a positive real number (a ≠ 1) and x is any positive real number, then the logarithmic function to the base a is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a}\ln x$$

<u>Derivatives for Bases other than e:</u>

Let a be a positive real number (a ≠ 1) and let u be a differentiable function of x.

1. $\frac{d}{dx}\left[a^x\right] = (\ln a)a^x$
2. $\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \frac{du}{dx}$
3. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}\frac{du}{dx}$

Ex: Find the derivative of each function

a. $y=5^x$ b. $y=x3^{x^2+1}$ c. $y=5\log_3 4x$

Ex: Evaluate $\int 5^x\,dx$.

Previously the power rule required n to be a rational number. However, now the rule can be extended to cover any real value number.

The Power Rule for Real Exponents

Let n be any real number and let u be a differentiable function of x.

$$\frac{d}{dx}\left[x^n\right]=nx^{n-1} \qquad \frac{d}{dx}\left[u^u\right]=nu^{n-1}\frac{du}{dx}$$

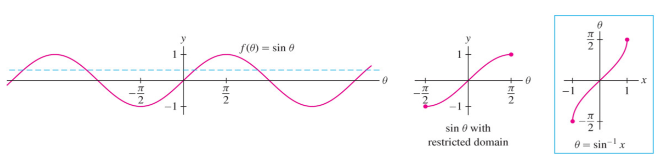
Ex: Find the derivative of each function

a. $y=x^{\sqrt{2}}$ b. $y=x^{2x^2-1}$

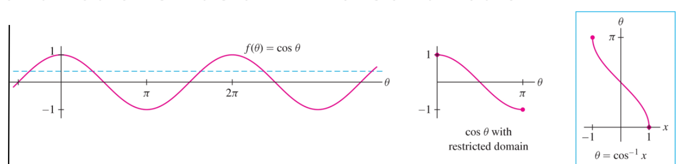
Inverse Trigonometric Functions: Differentiation

Objective: Develop properties of the six inverse trigonometric functions. Differentiate an inverse trigonometric function. Review the basic differentiation formulas for elementary functions.

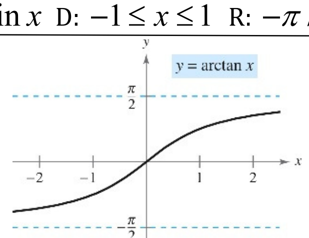
None of the six basic trigonometric functions has an inverse function!



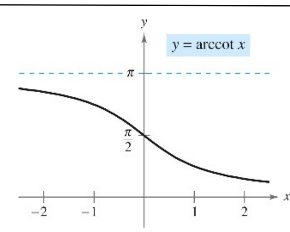
$y=\arcsin x$ D: $-1\le x\le 1$ R: $-\pi/2\le y\le \pi/2$



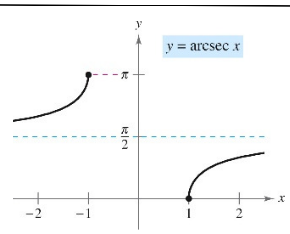
$y=\arccos x$ Domain: $-1\le x\le 1$ Range: $0\le y\le \pi$



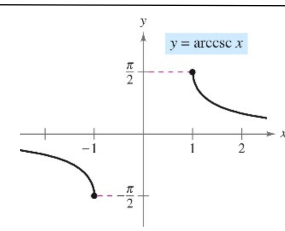
$y=\arctan x$ D: $-\infty\le x\le\infty$ R: $-\pi/2<y<\pi/2$



$y=\text{arccot}\, x$ D: $-\infty\le x\le\infty$ R: $0<y<\pi$



$y=\text{arcsec}\, x$ D: $|x|\ge 1$ R: $0<y<\pi, y\ne\pi/2$



$y=\text{arccsc}\, x$ D: $|x|\ge 1$ R: $-\pi/2\le y\le\pi/2, y\ne 0$

Ex: Evaluate: a. arcsin(- ½) b. arcos(0) c. $\arctan(\sqrt{3})$

Properties of Inverse Trigonometric Functions

1. If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\sin^{-1} x) = x \text{ and } \sin^{-1}(\sin y) = y.$$

2. If $-1 \le x \le 1$ and $0 \le y \le \pi$, then

$$\cos(\cos^{-1} x) = x \text{ and } \cos^{-1}(\cos y) = y.$$

3. If x is a real number and $-\pi/2 \le y \le \pi/2$, then

$$\tan(\tan^{-1} x) = x \text{ and } \tan^{-1}(\tan y) = y.$$

similar properties hold for other inverse trigonometric functions

Ex: Solve $\arctan(2x-3) = \pi/4$

Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}\left[\arcsin u\right] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\left[\arccos u\right] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[\arctan u\right] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}\left[\operatorname{arccot} u\right] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}\left[\operatorname{arcsec} u\right] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}\left[\operatorname{arccsc} u\right] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

To derive these formulas, you can use implicit differentiation

Ex: Differentiate:

a. $y = \arcsin(x^2)$ **b.** $f(x) = \arctan(3x+2)$ **c.** $y = \arccos\sqrt{x-1}$ **d.** $y = \operatorname{arcsec} e^{x^2}$

Ex: Differentiate $y = \arcsin x + x\sqrt{1-x^2}$

Inverse Trigonometric Functions: Integration

Objective: Integrate functions whose antiderivatives involve inverse trigonometric functions. Use completing the square to integrate a function. Review the basic integration formulas involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C \quad , \quad \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C \quad , \quad \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \text{arcsec} \frac{|u|}{a} + C$$

These rules all come from the preceding derivative rules of inverse functions. Since the arcsin and arccos derivatives are the negative of each other you only need one derivative for the pair.

Ex: Evaluate

a. $\int \frac{dx}{\sqrt{9 - x^2}}$

b. $\int \frac{dx}{5 + 16x^2}$

c. $\int \frac{dx}{x\sqrt{4x^2 - 9}}$

d. $\int \frac{dx}{\sqrt{e^{2x} - 1}}$

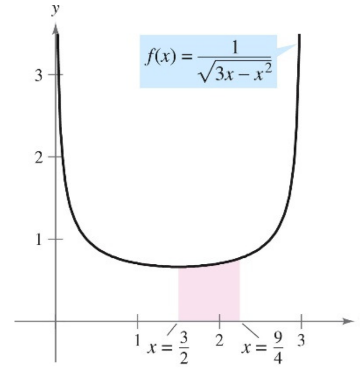
e. $\int \frac{x + 3}{\sqrt{4 - x^2}} dx$

Remember completing the square!?!?!? Well, you need it for Calculus too!

$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c = \left(x + \left(\frac{b}{2}\right)\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Ex: Evaluate $\int \frac{dx}{x^2 - 4x + 7}$

Ex: Find the area of the region bounded by the graph of $f(x) = \dfrac{1}{\sqrt{3x - x^2}}$ the x-axis, and the lines $x = 3/2$ and $x = 9/4$.



Ex: Find as many of the following integrals as you can using the formulas and techniques studied so far.

a. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - 1}}$

b. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2 - 1}}$

c. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 1}}$

d. $\displaystyle\int \frac{dx}{x \ln x}$

e. $\displaystyle\int \frac{\ln x\,dx}{x}$

f. $\displaystyle\int \ln x\,dx$