The Natural Logarithmic Function: Differentiation

Objective: Develop and use properties of the natural logarithmic function. Understand the definition of the number *e*. Find derivatives of functions involving the natural logarithmic function.

Recall $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t} dt \qquad x > 0$$

The domain of the natural logarithmic function is the set of all positive real numbers.

Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties.

1. The domain if $(0,\infty)$ and the range is $(-\infty,\infty)$.

2. The function is continuous, increasing, and one-to-one.

3. The graph is concave down.

Logarithmic Properties

If *a* and *b* are positive numbers and *n* is rational, then the following properties are true.

1. $\ln(1) = 0$

2. $\ln(ab) = \ln a + \ln b$

3. $\ln(a^n) = n\ln a$

4. $\ln(a / b) = \ln a - \ln b$

Ex: Expand

a. $\ln 5x^3\sqrt{3x+2}$ **b.** $\ln \frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}}$

The Number e

e is the base for the natural logarithm, e = 2.71828182..

Definition of e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t} dt = 1$$

The Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$ **2.** $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$

Ex: a. $\frac{d}{dx}[\ln(5x)]$ **b.** $\frac{d}{dx}[\ln(5x^2+2)]$ **c.** $\frac{d}{dx}[x^2 \ln x]$ **d.** $\frac{d}{dx}[(\ln x^2)^4]$

Ex: Differentiate $f(x) = \ln\sqrt{5x^2+2}$

Ex: Differentiate $f(x) = \ln\frac{x(2-x^2)^3}{\sqrt{5x^4-3}}$

Ex: Find the derivative of $y = \frac{(x-3)^2}{\sqrt{x^3+2}}, \quad x \neq 3$

Derivative Involving Absolute Value

If u is a differentiable function of x such that u ≠ 0, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

Ex: Find the derivative of $f(x) = \ln|\cos x|$

The Natural Logarithmic Function: Integration

Objective: Use the Log Rule for Integration to integrate a rational function.
Integrate trigonometric functions.

The differentiation rules: $\frac{d}{dx}[\ln|x|] = \frac{1}{x}$ and $\frac{d}{dx}[\ln|u|] = \frac{u'}{u}$

Log Rule for Integration

Let *u* be a differentiable function of x

1. $\int \frac{1}{x}\,dx = \ln|x| + C$ **2.** $\int \frac{1}{u}\,du = \ln|u| + C$

Because $du = u'dx$ the second formula can be written as

$\int \frac{u'}{u}\,dx = \ln|u| + C$ or $\int \frac{du}{u} = \ln|u| + C$

Ex: Evaluate

a. $\int \frac{2}{x}\,dx$ b. $\int \frac{1}{3x-2}\,dx$

Ex: Find the area of the region bounded by the graph $y = \frac{2x}{x^2+3}$, the x-axis, and the line x = 3.

Ex: **a.** $\int \frac{4x^3+1}{x^4+x}dx$ **b.** $\int \frac{\csc^2 x}{\cot x}dx$ **c.** $\int \frac{x+1}{x^2+2x}dx$ **d.** $\int \frac{1}{3x+2}dx$

e. $\int \frac{x^3+2x^2-x+1}{x^2+1}dx$ (use long division first) **f.** $\int \frac{2x}{(x+1)^2}dx$

Guidelines for Integration

1. Learn a basic list of integration formulas.

2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice for u that will make the integrand conform to the formula.

3. If you can't find a u-substitution that works, try altering the integrand. Try a trigonometric identity, multiplication or division. Be creative.

4. If you have access to computer software that will find the antiderivative symbolically, use it.

Ex: Find $\int \tan x dx$

Ex: Find $\int \sec x dx$

Integrals of the Six Basic Trig Functions

1. $\int \sin u du = -\cos u + C$ **2.** $\int \cos u du = \sin u + C$

3. $\int \tan u du = -\ln|\cos u| + C$ **4.** $\int \cot u du = \ln|\sin u| + C$

5. $\int \sec u du = \ln|\sec u + \tan u| + C$ **6.** $\int \csc u du = -\ln|\csc u + \cot u| + C$

Ex: Evaluate: $\int_0^{\pi/4} \sqrt{1+\tan^2 x}dx$

Exponential Functions: Differentiation and Integration

Objective: Develop properties of the natural exponential function. Differentiate natural exponential functions. Integrate natural exponential functions.

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function f(x) = lnx is called the natural exponential function and is denoted by

$f^{-1}(x) = e^x$

that is,

$y = e^x$ if and only if $x = \ln y$

Recall: $\ln(e^x) = x$ and $e^{\ln x} = x$

Ex: Solve $7 = e^{x+1}$

Remember your exponent rules!

Properties of the Natural Exponential Function

1. The domain of $f(x) = e^x$ is $(-\infty,\infty)$, and the range is $(0,\infty)$

2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.

3. The graph of $f(x) = e^x$ is concave upward on its entire domain.

4. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$

Derivative of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\frac{d}{dx}[e^x] = e^x$ **2.** $\frac{d}{dx}[e^u] = e^u \frac{du}{dx}$

Ex: a. $\frac{d}{dx}[e^{5x+3}]$ **b.** $\frac{d}{dx}[e^{(-3+x)/x}]$ **c.** $\frac{d}{dx}[x^3 e^{x^2}]$

Integration Rules for Exponential Functions

Let u be a differentiable function of x.

1. $\int e^x dx = e^x + C$ **2.** $\int e^u du = e^u + C$

Ex: Find $\int e^{5x+1} dx$

Ex: a. $\int \frac{e^{1/x}}{x^2} dx$ **b.** $\int \cos x e^{\sin x} dx$ **c.** $\int \frac{5x^2}{e^{x^3}} dx$ **d.** $\int \frac{e^{3x} - 3e^{2x} + e^x}{e^{2x}} dx$

Ex: Evaluate each definite integral

a. $\int_0^1 \frac{1}{e^x} dx$ **b.** $\int_0^1 \frac{e^x}{1+e^x} dx$ **c.** $\int_{-1}^0 [e^x \sin(e^x)] dx$