Laplace Transform

In elementary Calculus we learned that differentiation and integration transform one function into another function. These operation possess the linearity property. For example:
Let α and β be constants then

$$\frac{d}{dx}\left[\alpha f(x)+\beta g(x)\right]=\alpha\frac{d}{dx}f(x)+\beta\frac{d}{dx}g(x)$$

The same is true for integration, definite and indefinite.

Integral Transform

If f(x,y) is a function of two variables, then a definite integral of f with respect to one of the variables leads to a function of the other variable. Example: by holding y constant,

$\int_1^2 2xy^2dx = 3y^2$. Similarly, $\int_a^b K(s,t)f(t)dt$ transforms a function f of variable t into a function F of variable s.

We will be particularly interested in an **integral transform**, where the interval of integration is the unbounded interval [0,∞). If f(t) is defined for t ≥ 0, then the improper integral $\int_0^\infty K(s,t)f(t)dt$ is defined as a limit:

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b\to\infty}\int_0^b K(s,t)f(t)dt$$

If the limit exists then we say the integral exists or is **convergent**; if the limit does not exist, the integral does not exist and is **divergent**. The limit will, in general, exist for only certain values of the variable s.

The function K(s,t) is called the **kernel** of the transform/ the choice K(s,t) = e^{-st} as the kernel gives us an especially important integral transform.

Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

is said to be the **Laplace transform** of f, provided the integral converges.

We will denote the function being transformed with lower case letters and its corresponding Laplace transform with capital letters, example:

$\mathscr{L}$ { f(t) } = F(s) , $\mathscr{L}$ { g(t) } = G(s) , etc.

Ex: Evaluate $\mathscr{L}$ { 1 }

Ex: Evaluate $\mathscr{L}$ { t }

Ex: Evaluate $\mathscr{L}$ { e^{-3t} }

Ex: Evaluate $\mathscr{L}$ { sin2t }

$\mathscr{L}$ is a **linear transform**, in other words, for a linear combination of functions we can write

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]dt = \alpha\int_0^\infty e^{-st} f(t)dt + \beta\int_0^\infty e^{-st} g(t)dt$$

whenever both integrals converge for s > c.

Hence it follows

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathscr{L}\{f(t)\} + \beta\mathscr{L}\{g(t)\} = \alpha F(s) + \beta G(s)$$

Example: $\mathscr{L}\{1+5t\} = \mathscr{L}\{1\} + 5\mathscr{L}\{t\} = \frac{1}{s} + \frac{5}{s^2}$

So from the previous examples we can do

Ex: $\mathscr{L}\{4e^{-3t} - 10\sin 2t\}$

Transforms of Some Basic Functions

a. $\mathscr{L}\{1\} = \frac{1}{s}$

b. $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}},\ n = 1,2,3...$

c. $\mathscr{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$

d. $\mathscr{L}\{e^{at}\} = \frac{1}{s - a}$

e. $\mathscr{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$

f. $\mathscr{L}\{\sinh kt\} = \frac{k}{s^2 - k^2}$

g. $\mathscr{L}\{\cosh kt\} = \frac{s}{s^2 - k^2}$

Sufficient Conditions for Existence of $\mathscr{L}$\{f(t)\}

There are two definitions need for this:

Piecewise Continuous: each of the separate pieces of a piecewise function are continuous by themselves.

Exponential Order: A function f is said to be of **exponential order c** if the exists constants c, M > 0, and T > 0 such that $|f(t)| \le Me^{ct}$ for all t > T.

In short the graph of f over the interval [T,∞) does not grow faster than the graph Me^{ct}

Example: t, e^{-t}, and 2cost are all of exponential order c =1

Where e^{t^2} is not of exponential order.

Sufficient Conditions for Existence

If f is piecewise continuous on [0,∞) and of exponential order c, then $\mathscr{L}$\{ f(t) \} exists for s > c.

Ex: Evaluate $\mathscr{L}$\{ f(t) \} where $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3 \end{cases}$

Behavior of F(s) as s→∞

IF f is piecewise continuous on (0,∞) and of exponential order and F(s) = **L** \{ f(t) \}, then $\lim_{s\to\infty} F(s) = 0$