Diff.Eq. Test # 2 Problem Set Prof. G. Buthusiem

1. Given that $y = c_1 + c_2 \cos x + c_3 \sin x$ is a family of solutions of $y''' + y' = 0$, find a member of the family satisfying the initial conditions $y(\pi) = 0, y'(\pi) = 2, y''(\pi) = -1$.

2. If $y_1(x)$ is a given solution to the DE find another solution $y_2(x)$ using reduction of order:
$y'' + 9y = 0, \quad y_1 = \sin 3x$.

3. Find the general solution of $y'' - 36y = 0$

4. Find the general solution for $2y'' + 4y' + 3y = 0$

5. Solve using undetermined coefficients $y'' - 2y' + y = x^3 + 4x$

6. Find the general solution to $3y''' + 10y'' + 15y' + 4y = 0$

7. Solve using variation of parameters $y'' - 2y' + 2y = e^x \tan x$

8. Solve $y''' - 5y'' + 6y' = 2\sin x + 8$

9. Solve $x^2 y'' - xy' + y = 2x$

Diff. Eq. Test #2 Problem Set G. Bothusiew

(1.) $y = c_1 + c_2 \cos x + c_3 \sin x$

$y' = -c_2 \sin x + c_3 \cos x$

$y'' = -c_2 \cos x - c_3 \sin x$

For $y(\pi) = 0$

$0 = c_1 + c_2 \cos \pi + c_3 \sin \pi$

$0 = c_1 + c_2$ $\quad \therefore c_1 = -1$

For $y'(\pi) = 2$

$2 = -c_2 \sin(\pi) + c_3 \cos \pi$

$-2 = c_3$

For $y''(\pi) = -1$

$-1 = -c_2 \cos \pi - c_3 \sin \pi$

$-1 = c_2$

$\therefore y = 1 - \cos x - 2 \sin x$ is the particular Solution

2. $y'' + 9y = 0 \qquad y_1 = \sin 3x$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx = \sin 3x \int \frac{e^{-\int 0 dx}}{\sin^2 3x} dx = \sin 3x \int \csc^2 3x$$

$$y_2 = \sin 3x \cdot e^{-c} \int \csc^2 3x dx = \sin 3x \cdot c \cdot \left(\frac{-\cot 3x}{3}\right)$$

$$= \sin 3x \cdot \left(\frac{-\cos 3x}{\sin 3x}\right) = -\cos 3x$$

$\therefore y_2 = -\cos 3x$

general solution is $y = c_1 \sin 3x + c_2 \cos 3x$

3. $y'' - 36y = 0$ linear homogeneous w/ constant coefficients

$m^2 - 36 = 0$

$(m+6)(m-6) = 0$

$m = -6 \quad m = 6$ two distinc Real Roots

$\therefore y = c_1 e^{-6x} + c_2 e^{6x}$ is the general solution.

4. $2y'' + 4y' + 3y = 0$ linear homogeneous w/ constant coeff.

$2m^2 + 4m + 3 = 0$

$m = \dfrac{-4 \pm \sqrt{16-24}}{4} = \dfrac{-4 \pm \sqrt{8}}{4} = -1 \pm \dfrac{\sqrt{2}}{2} i$

$\therefore y = e^{-x}\left(c_1 \cos \frac{\sqrt{2}}{2}x + c_2 \sin \sqrt{2}/2\, x\right)$ is the general solution.

5. $y'' - 2y' + y = x^3 + 4x$

1. Solve $y'' - 2y' + y = 0$

$m^2 - 2m + 1 = 0$

$(m+1)(m-1) \longrightarrow y_c = c_1 e^x + c_2 x e^x$

2. $D^4 (x^3 + 4x) = 0$

$D^4 (D^2 - 2D + 1)y = D^4 (x^3 + 4x) = 0$

$m = 0$ multiplicity $= 4$ $\qquad m = 1$ multiplicity $= 2$

$y = \underbrace{c_1 e^x + c_2 x e^x}_{y_c} + \underbrace{c_4 + c_5 x + c_6 x^2 + c_7 x^3}_{y_p}$

3. Find c_4, c_5, c_6, c_7

$$y_p = c_4 + c_5 x + c_6 x^2 + c_7 x^3$$

$$y_p' = c_5 + 2c_6 x + 3c_7 x^2$$

$$y_p'' = 2c_6 + 6c_7 x$$

plug into $y'' - 2y' + y = x^3 + 4x$

$$2c_6 + 6c_7 x - 2c_5 - 4c_6 x - 6c_7 x^2 + c_4 + c_5 x + c_6 x^2 + c_7 x^3 =$$

$$c_7 x^3 + (c_6 - 6c_7)x^2 + (6c_7 - 4c_6 + c_5)x + (2c_6 - 2c_5 + c_4) =$$

$c_7 = 1$ $\quad c_6 - 6c_7 = 0$ $\quad 6c_7 - 4c_6 + c_5 = 4$ $\quad 2c_6 - 2c_5 + c_4 = 0$

$\therefore c_6 = 6$ $\quad \therefore c_5 = 22$ $\quad \therefore c_4 = 32$

$$y_p = 32 + 22x + 6x^2 + x^3$$

$$\therefore y = y_c + y_p = c_1 e^x + c_2 x e^x + 32 + 22x + 6x^2 + x^3$$

6. $3y''' + 10y'' + 15y' + 4y = 0$ linear homogeneous w/ constant coef.

$$3m^3 + 10m^2 + 15m + 4 = 0$$

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & 10 & 15 & 4 \\ & & -1 & -3 & -4 \\ \hline & 3 & 9 & 12 & \boxed{0} \end{array}$$

$$(m + 1/3)(3m^2 + 9m + 12) = 0$$

$$(3)(m^2 + 3m + 4) = 0$$

$m_1 = -1/3$ $\quad m_2 = \dfrac{-3 \pm \sqrt{9 - 16}}{2} = \dfrac{-3 \pm \sqrt{-7}}{2} = \dfrac{-3}{2} \pm \dfrac{\sqrt{7}}{2} i$

$$y = c_1 e^{-1/3 x} + e^{-3/2 x}\left(c_2 \sin \frac{\sqrt{7}}{2} x + c_3 \cos \frac{\sqrt{7}}{2} x\right)$$

7. $y'' - 2y' + 2y = e^x \tan x$

$m^2 - 2m + 2 = 0$

$m = \frac{+2 \pm \sqrt{4-8}}{2} = 1 \pm i \quad \longrightarrow \quad y_c = e^x(c_1 \cos x + c_2 \sin x)$

$y_1 = e^x \cos x \qquad y_2 = e^x \sin x \qquad f(x) = e^x \tan x$

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x \cos x - e^x \sin x & e^x \sin x + e^x \cos x \end{vmatrix} = e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & e^x \sin x \\ e^x \tan x & e^x \sin x + e^x \cos x \end{vmatrix} = -e^{2x} \sin x \tan x$$

$$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x \cos x - e^x \sin x & e^x \tan x \end{vmatrix} = e^{2x} \cos x \tan x$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{2x} \sin x \tan x}{e^{2x}} = -\sin x \tan x = \frac{-\sin^2 x}{\cos x} = \frac{-(1-\cos^2 x)}{\cos x}$$

$$= \cos x - \sec x$$

$$u_2' = \frac{W_2}{W} = \frac{e^{2x} \cos x \tan x}{e^{2x}} = \cos x \tan x = \sin x$$

$u_1 = \int (\cos x - \sec x)\,dx = \sin x - \ln|\sec x + \tan x|$

$u_2 = \int \sin x\,dx = -\cos x$

$y_p = u_1 y_1 + u_2 y_2 = e^x \cos x(\sin x - \ln|\sec x + \tan x|) - e^x \sin x \cos x$

7. Continued

$$y = y_c + y_p$$
$$= e^x(c_1 \cos x + c_2 \sin x) + e^x \cos x \ln|\sec x + \tan x|$$

8. $y''' - 5y'' + 6y' = 2\sin x + 8$

$y''' - 5y'' + 6y' = 0$

$m^3 - 5m^2 + 6m = 0$

$m(m-3)(m-2) = 0$

$m = 0 \quad m = 3 \quad m = 2$

$y_c = c_1 + c_2 e^{3x} + c_3 e^{2x}$

$(D^2+1)D(2\sin x + 8) = 0$

$(D^2+1)D(D^3 - 5D^2 + D) = 0$

$D = 0 \quad D = 3 \quad D = 2 \quad D = \pm i$

mult = 2

y_p

$y = c_1 + c_2 x + c_3 e^{3x} + c_4 e^{2x} + c_5 \cos x + c_6 \sin x$

y_c

$y_p = Ax + B\cos x + C\sin x$

$y_p' = A - B\sin x + C\cos x$

$y_p'' = -B\cos x - C\sin x$

$y_p''' = +B\sin x - C\cos x$

∴ plug into $y''' - 5y'' + 6y' = 2\sin x + 8$

$B\sin x - C\cos x$

$5C\sin x + 5B\cos x$

$-6B\sin x + 6C\cos x \quad + 6A$

$(5C - 5B)\sin x + (5B + 5C)\cos x + 6A = 2\sin x + 8$

$5C - 5B = 2 \qquad 5B + 5C = 0 \qquad 6A = 8$

$A = 4/3$

$10C = 2$

$C = 1/5 \rightarrow B = -1/5$

$\therefore\ y = y_c + y_p$

$$= c_1 + c_2 e^{3x} + c_3 e^{2x} + \frac{4}{3}x - \frac{1}{5}\cos x + \frac{1}{5}\sin x$$

or

$$y = c_1 + c_2 e^{3x} + c_3 e^{2x} + 20x + 9\cos x + 3\sin x$$

9. $x^2y'' - xy' + y = 2x$ 1st Solve $x^2y'' - xy' + y = 0$

$a=1 \quad b=-1 \quad c=1$ Aux. eq. $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \quad m = 1$ repeated

$$y_c = c_1 x + c_2 x \ln x$$

$$y_1 = x \quad y_2 = x \ln x$$

Now Solve

$$x^2y'' - xy' + y = 2x \longrightarrow y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{2}{x}$$

$$f(x) = 2/x \qquad y_p = u_1 y_1 + u_2 y_2$$

$$u_1' = \frac{-y_2 f(x)}{W} = \frac{-x\ln x(2/x)}{W} = -\frac{2\ln x}{x}$$

$$u_2' = \frac{y_1 f(x)}{W} = \frac{x \cdot 2/x}{W} = \frac{2}{x}$$

$$W = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x(\ln x + 1) - x\ln x = x$$

$$u_1 = \int u_1' = \int -\frac{2\ln x}{x}\,dx = -(\ln x)^2$$

$$u_2 = \int u_2' = \int \frac{2}{x}\,dx = 2\ln|x|$$

$$y_p = -x(\ln x)^2 + 2x(\ln x)^2 = x\ln x$$

$$y = y_c + y_p = c_1 x + c_2 x\ln x + x\ln x$$