Diff.Eq. Test # 1 Problem Set

G.Buthusiem

1. If $y = c_1 \cos t + c_2 \sin t$ is a two – parameter family of solutions of the second order DE $y'' + y = 0$, find the solution of the second order IVP of this DE given $y\left(\frac{\pi}{2}\right) = 0$ and $y'\left(\frac{\pi}{2}\right) = 1$

2. Solve the separable DE: $\frac{dy}{dx} + 2xy^2 = 0$

3. Solve the linear DE: $3\frac{dy}{dx} + 12y = 4$

4. Solve the homogeneous DE: $(x+y)dx + xdy = 0$

5. Solve the exact DE: $(\sin y - y\sin x)dx + (\cos x + x\cos y - y)dy = 0$

6. Solve: $\frac{y}{x}\frac{dy}{dx} = \frac{e^x}{\ln y}$

7. Solve: $xyy' = 3y^2 + x^2$ subject to y(-1) = 2

8. Solve: $(x^2+4)\frac{dy}{dx} = 2x - 8xy$

9. Solve: $x\frac{dy}{dx} + 4y = x^4 y^2$

Also look at pg 73 #5 a, b, d, e, f ,g, h, I, j, k, l, m, n

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1. If $y = c_1\cos t + c_2 \sin t$ is a two – parameter family of solutions of the second order DE $y'' + y = 0$, find the solution of the second order IVP of this DE given $y\left(\frac{\pi}{2}\right) = 0$ and $y'\left(\frac{\pi}{2}\right) = 1$

$y = c_1\cos t + c_2 \sin t$

$y(\pi/2) = 0$

$0 = c_1\cos\pi/2 + c_2\sin\pi/2$

$0 = c_2$

$y' = -c_1\sin t + c_2\cos t$

$y'(\pi/2) = 1$

$1 = -c_1\sin\pi/2 + c_2\cos\pi/2$

$1 = -c_1$

$-1 = c_1$

$y = -\cos t$ solves the IVP for $y'' + y = 0$

2. Solve the separable DE: $\dfrac{dy}{dx} + 2xy^2 = 0$

$$\frac{dy}{y^2} = -2x\,dx$$

$$\int \frac{dy}{y^2} = \int -2x\,dx$$

$$-\frac{1}{y} = -\frac{x^2}{2} + C$$

$$-\frac{1}{y} = -x^2 + C$$

$$\boxed{y = \frac{1}{x^2 - C}}$$

3. Solve the linear DE: $3\frac{dy}{dx}+12y=4$

$$\frac{dy}{dx}+4y=\frac{4}{3}$$

$P(x)=4 \quad f(x)=4/3$

$$\mu(x)=e^{\int 4dx}=e^{4x}$$

$$e^{4x}y=\int e^{4x}\cdot\frac{4}{3}\,dx+C$$

$$e^{4x}y=\frac{1}{3}e^{4x}+C$$

$$\boxed{y=\frac{1}{3}+ce^{-4x}}$$

4. Solve the homogeneous DE: $(x+y)dx+xdy=0$

let $y=ux \quad dy=udx+xdu$

$$(x+ux)dx+x(udx+xdu)=0$$

$$x(1+u)dx+xudx+x^2du=0$$

$$(1+2u)dx+xdu=0$$

$$\frac{1}{x}dx+\frac{1}{1+2u}du=0$$

$$\int\frac{1}{x}dx+\int\frac{1}{1+2u}du=0$$

$$\ln|x|+\frac{1}{2}\ln|1+2u|=C$$

$$\ln\left|(x)\sqrt{1+2u}\right|=C$$

$$x\sqrt{1+2(y/x)}=e^C$$

$$x\frac{\sqrt{x+2y}}{\sqrt{x}}=C$$

$$\boxed{\sqrt{x(x+2y)}=C}$$

5. Solve the exact DE: $\underbrace{(\sin y - y\sin x)}_{M(x,y)}dx + \underbrace{(\cos x + x\cos y - y)}_{N(x,y)}dy = 0$

$$\left.\begin{aligned} M_y &= \cos y - \sin x \\ N_x &= -\sin x + \cos y \end{aligned}\right\} \text{equal} \therefore \text{exact}$$

if $M(x,y) = \frac{\partial f}{\partial x}$ then $\int (\sin y - y\sin x)dx = f(x,y)$

$= x\sin y + y\cos x + g(y) = f(x,y)$

if $N(x,y) = \frac{\partial f}{\partial y}$ then $\frac{\partial}{\partial y}[x\sin y + y\cos x + g(y)] = \cos x + x\cos y - y$

$x\cos y + \cos x + g'(y) = \cos x + x\cos y - y$

$\therefore g'(y) = -y$ & $g(y) = -\frac{y^2}{2}$

$f(x,y) = x\sin y + y\cos x - \frac{y^2}{2}$

& $\boxed{x\sin y + y\cos x - y^2/2 = c}$ is a solution to the DE

6. Solve: $\frac{y}{x}\frac{dy}{dx} = \frac{e^x}{\ln y}$ this is seperable

$y\ln y\,dy = xe^x dx$

$\int y\ln y\,dy = \int xe^x dx$

$u = \ln y$ $dv = y\,dy$ $\quad r = x$ $ds = e^x dx$

$du = \frac{1}{y}dy$ $v = \frac{y^2}{2}$ $\quad dr = dx$ $s = e^x$

$\frac{y^2}{2}\ln y - \int \frac{y}{2}dy = xe^x - \int e^x dx$

$\frac{y^2}{2}\ln y - \frac{y^2}{4} = xe^x - e^x + c$

$\frac{y^2}{4}(2\ln y - 1) = (x-1)e^x + c$

7. Solve: $xyy' = 3y^2 + x^2$ subject to y(-1) = 2

$$xy\frac{dy}{dx} = 3y^2+x^2$$

$$0 = (3y^2+x^2)dx - xy\,dy \quad \text{Homogeneous}$$

let $y = ux \qquad dy = u\,dx + x\,du$

$$0 = (3u^2x^2+x^2)dx - ux^2(u\,dx + x\,du)$$

$$= (3u^2+1)dx - u^2 dx - ux\,du$$

$$= (2u^2+1)dx - ux\,du$$

$$= \frac{dx}{x} - \frac{u\,du}{2u^2+1}$$

$$c = \int\frac{dx}{x} - \int\frac{u\,du}{2u^2+1}$$

$$C = \ln|x| - \frac{1}{4}\ln|2u^2+1|$$

$$\to C = \ln\left|\frac{x^6}{2y^2+x^2}\right|$$

$$\boxed{C = \frac{x^6}{2y^2+x^2}} \quad \text{Family of solutions}$$

using $y(-1)=2$

$$C = \frac{1}{9} \longrightarrow \boxed{\frac{1}{9} = \frac{x^6}{2y^2+x^2}} \quad \text{Particular Solution to IVP}$$

8. Solve: $(x^2+4)\dfrac{dy}{dx} = 2x - 8xy$

$$\frac{dy}{dx} + \frac{8x}{x^2+4}y = \frac{2x}{x^2+4} \quad \text{linear}$$

$$P(x) = \frac{8x}{x^2+4} \qquad f(x) = \frac{2x}{x^2+4}$$

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{8x}{x^2+4}dx} = e^{4\ln|x^2+4|} = (x^2+4)^4$$

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}f(x)\,dx + c$$

$$(x^2+4)^4 y = \int (x^2+4)^4 \cdot \frac{2x}{x^2+4}\,dx + C$$

$$(x^2+4)^4 y = \int (x^2+4)^3\, 2x\,dx + C$$

$$(x^2+4)^4 y = \frac{(x^2+4)^4}{4} + C$$

$$\boxed{y = \frac{1}{4} + \frac{C}{(x^2+4)^4}}$$

9. Solve: $x\dfrac{dy}{dx}+4y=x^4y^2$ Bernoulli

$$y^{-2}\frac{dy}{dx}+\frac{4}{x}y^{-1}=x^3$$

y^{-n} $n=2$

let $w=y^{1-n}$

$w=y^{-1}$

$$\frac{dw}{dx}=-y^{-2}\frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx}=-y^{2}\frac{dw}{dx}$$

$$y^{-2}\left(-y^2\frac{dw}{dx}\right)+\frac{4}{x}w=x^3$$

$$\frac{dw}{dx}-\frac{4}{x}\cdot w=-x^3$$ Now its linear

$P(x)=-\frac{4}{x}$ $f(x)=-x^3$

$$\mu(x)=e^{\int P(x)dx}=e^{\int -4/x\,dx}=e^{-4\ln x}=x^{-4}$$

$$x^{-4}w=\int x^{-4}(-x^3)dx+C$$

$$x^{-4}w=-\ln x+C$$

$$w=-x^4\ln x+Cx^4$$

Now use $w=y^{-1}$

$$\boxed{y=(-x^4\ln x+Cx^4)^{-1}}$$