Solving Systems of Linear DE by Elimination

Solution of a System: A **solution** of a system of differential equations is a set of differentiable functions $x = f(t)$, $y = g(t)$, $z = h(t)$, and so on that satisfies each equation of the system on some interval I.

Systematic Elimination: the elimination of an unknown in a system of linear differential equations is expedited by rewriting each equation on the system in differential operator notation

Recall

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = g(t)$$

where the a_i, i = 0,1…,n are constants can be written as

$$(a_n D^n + a_{n-1} D^{(n-1)} + \ldots + a_1 D + a_0) y = g(t)$$

If the nth order differential operator factors into differential operators of lower order, then the factors commute.

Ex:

$$x'' + 2x' + y'' = x + 3y + \sin t$$

$$x' + y' = -4x + 2y + e^{-t}$$

Can be rewritten as

$$\begin{matrix} x'' + 2x' - x + y'' - 3y = \sin t \\ x' + 4x + y' - 2y = e^{-t} \end{matrix} \rightarrow \begin{matrix} (D^2 + 2D - 1)x + (D^2 - 3)y = \sin t \\ (D - 4)x + (D - 2)y = e^{-t} \end{matrix}$$

Method of Solution:

Consider the system of linear first order DE

$$\begin{matrix} \dfrac{dx}{dt} = 3y \\ \dfrac{dy}{dt} = 2x \end{matrix} \quad \text{or equivalently} \quad \begin{matrix} Dx - 3y = 0 \\ 2x - Dy = 0 \end{matrix}$$

- We can eliminate y by operating the first equation by D and multiplying the second by -3.
- We can solve the resulting auxiliary equation and get a solution for x(t)
- We could also go back and eliminate x and find a solution for y(t)
- The next step is to find what values of c_1, c_2, c_3, c_4 satisfy the system. Plug x(t) and y(t) back into one of the original equations and simplify the number of parameters.

Ex: Solve $\begin{aligned} Dx+(D+2)y=0 \\ (D-3)x-2y=0 \end{aligned}$

Ex: Solve $\begin{aligned} x'-4x+y''=t^2 \\ x'+x+y'=0 \end{aligned}$

Using Determinants:

Symbolically if L_1, L_2, L_3, and L_4 denote linear differential operators with constant coefficients, then a system of linear differential equations in two variables x and y can be written as

$$\begin{aligned} L_1x+L_2y=g_1(t) \\ L_3x+L_4y=g_2(t) \end{aligned}$$

Eliminating variables we would get

$$(L_1L_4-L_2L_3)x=f_1(t) \ \ and \ \ (L_1L_4-L_2L_3)x=f_2(t)$$

Where

$$f_1(t)=L_4g_1(t)-L_2g_2(t) \ \ and \ \ f_2(t)=L_1g_2(t)-L_3g_1(t)$$

If we use Cramers rule:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x=\begin{vmatrix} g_1 & L_2 \\ g_2 & L_4 \end{vmatrix} \ \ and \ \ \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y=\begin{vmatrix} L_1 & g_1 \\ L_3 & g_2 \end{vmatrix}$$

The left hand determinant in each equation can be expanded in the usual sense. However the right hand one need a little more attention. We need to exand the determinant by actually have the differential operators actually operating on the functions g_1 and g_2.

If $\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} \neq 0$ and is a differential operator of order n, then

- The system can be uncoupled into two nth order differential equations in x and y
- The characteristic equation and hence the complimentary function of each of these differential operators are the same
- Since x and y both contain n constants, there are a total of 2n contants appearing
- The total number of independent constants in the solution of the system in n

If $\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} = 0$

then, the system may have a solution containing any number of independent constants or may have no solution at all.

Ex: Solve $\begin{aligned} x'=3x-y-1 \\ y'=x+y+4e^t \end{aligned}$