Sequences and Series

Sequence:
An ordered list of numbers, called **terms**. The terms of a sequence are often arranged in a pattern.

- $a_1, a_2, a_3, \ldots a_n$ are the **terms** of the sequence and
- a_n is the **nth term**

A sequence is usually defined as a function whose domain is the set of positive integers but, it's easier to denote a sequence in subscript form rather than function notation.

Infinite Sequence:
Has three dots (**…** → called an ellipsis) at its end. This indicates that the sequence goes on forever, or continues without end.

Explicit Formula (general term):
A formula that defines the ***n****th* term, or general term, of a sequence; with an explicit formula, each term of the sequence can be found by substituting the number of the term for ***n***.

Expressing a Sequence
a. $\{ a_n \} = \{3 + (-1)^n\}$ **b.** $\{ b_n \} = \{\frac{n}{1-2n}\}$

c. a **recursively defined** sequence $\{c_n\}$, where $c_1 = 25$ and $c_{n+1} = c_n - 5$.

Ex: Find the explicit form of a sequence whose first four terms are

a. $\{2,4,6,8,\ldots\}$ **b.** $\left\{1,-\frac{1}{2},\frac{1}{4},-\frac{1}{8}\ldots\right\}$

Series:
If $a_1, a_2, a_3, \ldots a_n, \ldots$ is a sequence, the the expression

$$a_1 + a_2 + a_3 + \ldots a_n + \ldots$$

is called a **series.**

If the sequence is finite then the corresponding series is finite if the sequence is infinite then corresponding series is infinite.

Summation Notation:
Also called **Sigma Notation**, is a way to express a **series** in abbreviated form. It is denoted by the Greek letter sigma, **Σ**

$\sum_{k=1}^{4} a_k = a_1 + a_2 + a_3 + a_4$ k is the index of the series

Ex: Find the sum $\sum_{k=1}^{4} 2k+3$

Summation Properties:

For sequences a_k and b_k and positive integer n:

1. $\sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$
2. $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Summation Formulas:

For all positive integers n:

1. **Constant Series:** $\sum_{k=1}^{n} c = nc$
2. **Linear Series:** $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
3. **Quadratic Series:** $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Ex: Find the sum

a. $\sum_{k=1}^{7} 3$ **b.** $\sum_{k=1}^{5} k$ **c.** $\sum_{k=1}^{4} 2k+3$ **d.** $\sum_{k=1}^{100} k^2+k+1$

Arithmetic and Geometric Sequences and Series

Arithmetic Sequence:

Sequence whose successive terms differ by the same number, ***d***, called the **common difference**.

$$a_n - a_{n-1} = d \text{ or } a_n = a_{n-1} + d$$

n^{th} Term of an Arithmetic Sequence:

The general term, a_n, of an arithmetic sequence whose **1st** term is a_1 and whose common difference is ***d*** is given by the explicit formula

$$a_n = a_1 + d(n - 1)$$

Geometric Sequence:

A sequence in which the ratio of successive terms is the same number, ***r***, called the **common ratio**.

$$\frac{a_n}{a_{n-1}} = r \text{ or } a_n = ra_{n-1}$$

n^{th} Term of a Geometric Sequence:
The n^{th} term, a_n, of a geometric sequence whose 1st term is a_1 and whose common ration is r is given by the explicit formula $a_n = a_1 \bullet r^{n-1}$, where **n ≥ 1**.

Ex: Label the following as a geometric, arithmetic sequences, both, or neither. If geometric or arithmetic write the formula for the nth term.

a. {2, 3, 5, 7, 9, 10…} **b.** {-5, -3, -1, 1, …} **c.** {5, 5, 5, 5,…}
d. {2, -4, 8, -16,…}

Arithmetic Series:
The indicated **sum** of the terms of an arithmetic sequence.

Sum of the First *n* Terms of a finite Arithmetic Series:
The sum, S_n, of the first n terms of an arithmetic series with a first term of a_1 & n^{th} term a_n is given by:

$$\mathbf{S_n} = \mathbf{n}\left(\frac{a_1 + a_n}{2}\right)$$

Ex: Find the sum of the first 52 terms of an arithmetic series if the first term is 23 and the common difference is -2.

Ex: Find the sum of all the even numbers between -23 and 53.

Geomtric Series:
The indicated **sum** of the terms of an geometric sequence.

Sum of the 1st *n* Terms of a finite Geometric Series:
The sum, S_n, of the 1st n terms of a geometric series is given by:

$$\mathbf{S_n} = a_1\left(\frac{\mathbf{1 - r^n}}{\mathbf{1 - r}}\right)$$

where a_1 is the 1st term, r is the common ratio, $r \neq 1$.

Ex: Find the sum of the first 15 terms of a geometric series whose first term is 1 and r = 3.

Ex: Find the sum of $\sum_{k=1}^{7} 2(3^k)$

Sum of an INFINITE Geometric Series where a_1 is the 1st term, r is the common ratio and $|r|$ **< *1***.

$$S_n = \frac{a_1}{1-r}$$

If $|r| \geq 1$ then the infinite geometric series diverges *(the sum is infinite).*

Ex: Find the sum of $\sum_{k=1}^{\infty} 2(3^k)$