Rational Functions

Section Objectives: Students will know how to determine the domains, find the asymptotes, and sketch the graphs of rational functions.

A **rational function** is a function of the form

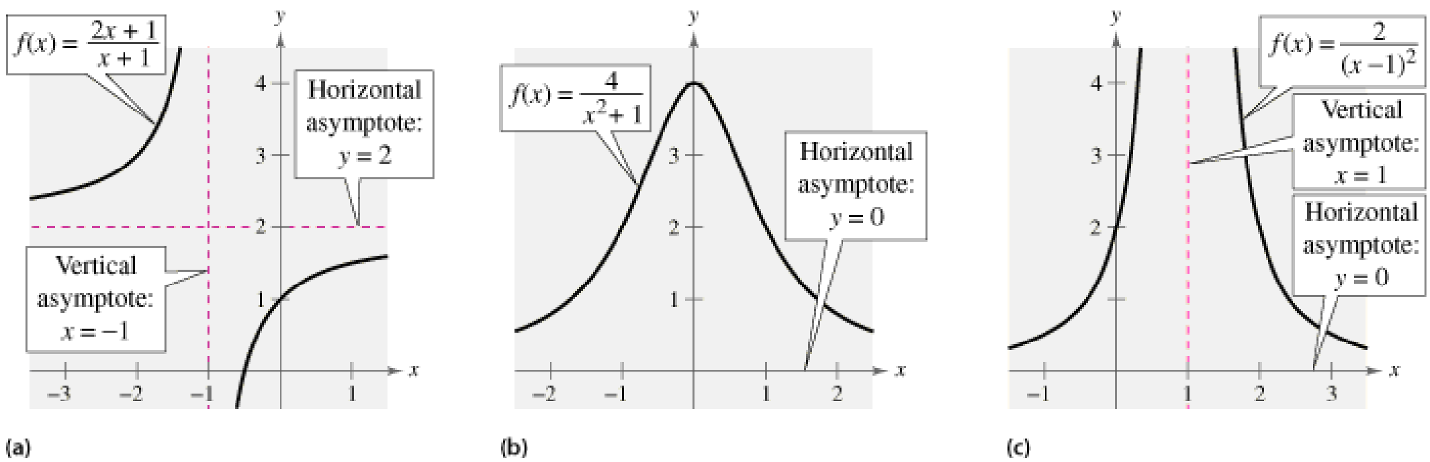
$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} \ldots + b_1 x + b_0}$$

where ***N(x)*** and ***D(x)*** are both polynomials. The domain of ***f (x)*** is all ***x*** such that ***D*(*x*)** $\neq$ **0**.

Ex: Find the domain of $f(x) = \frac{1}{x}$.

Definitions of Asymptotes

- The line *y* = *b* is a **horizontal asymptote** of the graph of ***f(x)*** if $f(x) \to b$ as $x \to -\infty$ or $x \to \infty$
- The line *x* = *a* is a **vertical asymptote** of the graph of ***f(x)*** if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$, either from the right or from the left.



Rules for Asymptotes of Rational Functions.

Let ***f(x)*** be a rational function given by

$$f(x)=\frac{N(x)}{D(x)}=\frac{a_n x^n + a_{n-1}x^{n-1}\ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1}\ldots + b_1 x + b_0}$$

where **N(x)** and **D(x)** have no common factors

1. The graph of ***f(x)*** has ***vertical*** **asymptote's** at the zeros of ***D(x)***

- *if N(x) and D(x) have common factors then f(x) will only have vertical asymptotes at the x values that are zeros of D(x) but not N(x). If N(x) and D(x) have common zeros then this will result in a different kind of discontinuity, a hole in the graph.*

2. The graph of *f* has **one *horizontal* asymptote** or **no horizontal asymptote**, depending on the degree of ***N(x)*** and ***D(x)***. Let n be the degree of N(x) and m be the degree of D(x).

a. If $n < m$, then $y = 0$ is the **horizontal asymptote** of the graph of ***f(x)***.

b. If $n = m$, then $y = a_n / b_m$ is the **horizontal asymptote** of the graph of ***f(x)***.

c. If $n > m$, then there is **no horizontal asymptote** of the graph of ***f(x)***.

Ex: Find any horizontal and vertical asymptotes of the following.

a) $f(x)=\frac{2x}{3x^2+1}$ **b)** $f(x)=\frac{2x^2}{x^2-1}$ **c)** $f(x)=\frac{x^2+1}{x}$

Analyzing Graphs of Rational Functions

Let $f(x)=\dfrac{N(x)}{D(x)}$, where **N(x)** and **D(x)** are polynomial with no common factors.

1. **Find** the **zero's** of the **denominator** (if any) by solving the equation **D(x) = 0**, then sketch the corresponding **vertical asymptotes.**
2. **Find** and **sketch** the **horizontal asymptote** (if any) by using the rules for finding the horizontal asymptote of a rational function.
3. **Find** and **plot** the **y-intercept** (if any) by evaluating **f(0)**.
4. **Find** the **zero's** of the **numerator** (if any) by solving the equation **N(x) = 0**, then plot the corresponding **x – intercepts**.
5. **Plot check points,** at least one point between and one point beyond each x – intercept and vertical asymptote.
6. Use smooth curves to **complete the graph** between and beyond the vertical asymptote.

Sketch the graph of the following rational functions

a) $f(x)=\dfrac{3}{x-2}$ **b)** $g(x)=\dfrac{2x-1}{x}$ **c)** $h(x)=\dfrac{x}{x^2-x-2}$

d) $a(x)=\dfrac{2(x^2-9)}{x^2-4}$

Slant Asymptotes (oblique asymptotes)

If the degree of the numerator is exactly one more than the degree of the denominator, then the graph has a

slant (or oblique) asymptote

To find the slant asymptote: perform the division of the rational function and find the quotient. Set the quotient equal to y and this will give you the equation of the slant asymptote.

Ex: Find the Slant Asymptote of $f(x)=\dfrac{x^2-x}{x-1}$

Reasoning for this procedure: *As* $x\to\infty$ *or* $x\to-\infty$ *the remainder goes to zero and the graph approaches the quotient.*

Ex: Graph $f(x)=\dfrac{x^2-x-2}{x-1}$

What if the numerator and denominator do have common factors?
You should:

- cancel out those common factors and graph following the procedure that was described earlier for graphing simplified rational functions
- then go back and set the factors that were canceled out of the original rational function equal to zero. Where those x-values would hit the graph will be represented by holes in the graph.

Ex: $f(x)=\dfrac{x-2}{x^2-4}$