Review of Power Series, Power Series Solutions

A power series in x - a is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + (x-a)^2 + \ldots$$

We also call this a power series centered at a.

Ex. $\sum_{n=0}^{\infty} (x+1)^n$ is centered at a = -1.

We are mainly going to be concerned with power series in x, such as

$\sum_{n=0}^{\infty} 2^{n-1} x^n = x + 2x^2 + 4x^3 + \ldots.$ that are centered at a = 0.

Important facts about power series:

• **Convergence:** a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is convergent at a specified value of x if its sequence of partial sums $\{s_n(x)\}$ converges.

• **Interval of Convergence:** every power series has an interval of convergence. The interval of convergence is the set of all real numbers x for which the series converges.

• **Radius of Convergence:** every power has a radius of convergence R and there are three possibilities:

I. The series only converges at its center, R=0

II. The series converges for all x satisfying |x – a| **< R** and diverges if |x – a|**>R**

III. The series converges for all x, R=∞

Recall that the absolute value inequality |x – a| < R is equivalent to the simultaneous inequality **a - R <x < a +R**. A power series might or might not converge at the endpoints **a - R** and **a + R** of this interval.

• **Absolute Convergence:** within its interval of convergence a power series converges absolutely. In other words, if x is a number in the interval of convergence and is not an endpoint of the interval, then the series of absolute values $\sum_{n=0}^{\infty} |c_n (x-a)^n|$ converges.

• **Ratio Test**: Convergence of a power series can often be determined by the ratio test. Suppose that $c_n \neq 0$ for all n and that $\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L$

If L<1, the series converges absolutely; if L>1 the series diverges; and of L=1, the test is inconclusive.

Ex: Find the interval of convergence for $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n}$

(many times Root Test can also be applied)

• **A Power Series Defines a Function:** A power series defines a function $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$ whose domain is the interval of convergence of the series. If the radius of convergence is R>0, then f is continuous, differentiable, and integrable on the interval (a-R,a+R). Moreover, $f'(x)$ and $\int f(x)dx$ can be found by term-by-term differentiation and integration. Convergence at an endpoint may be either lost by differentiation or gained through integration. If $y=\sum_{n=0}^{\infty} c_n x^n$ is a power series in x, then the first two derivatives are $y'=\sum_{n=0}^{\infty} nc_n x^{n-1}$ and $y''=\sum_{n=0}^{\infty} n(n-1)c_n x^{n-2}$ Notice that the first term in the first derivative and the first two terms in the second derivative are zero. We omit these zero terms and write $y'=\sum_{n=0}^{\infty} nc_n x^{n-1}$ and $y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$

• **Identity Property**: If $f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n = 0$ R > 0 for all number x in the interval of convergence, then c_n =0 for all n.

• **Analytic at a point**: A function is analytic at a point a if it can be represented by a power series in x-a with positive or infinite radius of convergence. In calculus it is seen that the functions such that e^x, cosx, sinx, ln(1-x), and so on can be represented by the Taylor series. Recall

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+...,$$
$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+... for\,|x|<\infty$$
$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+...$$

Taylor series centered at 0 called Maclaurin Series.

• **Arithmetic of Power Series:** Power series can be combined through the operations of addition, multiplication and division. The procedures for power series are similar to those by which two polynomials are added, multiplied, and divided - that is, we add coefficients of like powers of x, use the distributive law and collect like terms, and perform long division.

Ex:

$$e^x \sin x = \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\ldots\right)$$

$$= (1)x+(1)x^2+\left(-\frac{1}{6}+\frac{1}{2}\right)x^3+\left(-\frac{1}{3!}+\frac{1}{6}\right)x^4+\left(\frac{1}{5!}-\frac{1}{12}+\frac{1}{24}\right)x^5+\ldots$$

$$= x+x^2+\frac{x^3}{3}-\frac{x^5}{30}\ldots$$

Ex: Write $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}+\sum_{n=0}^{\infty} c_n x^{n+1}$ as a single power series whose general term involves x^k.

<u>**Power Series Solution of a Differential Equation**</u>

We can see that $y=e^{x^2}$ is a solution to $\frac{dy}{dx}-2xy=0$

By replacing x by x^2 in the Maclaurin series for e^x we can write the solution as

$$y=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!} \text{ which converges everywhere}$$

We can then find a way to find this solution directly using a method very similar to the technique of undetermined coefficients.

Ex: Find a solution to $\frac{dy}{dx}-2xy=0$

Assume a solution of the form $y=\sum_{n=0}^{\infty} c_n x^n$ exists then

$$y'=\sum_{n=1}^{\infty} nc_n x^{n-1}$$

So

$$\frac{dy}{dx}-2xy=\sum_{n=1}^{\infty} nc_n x^{n-1}-\sum_{n=0}^{\infty} 2c_n x^{n+1} = 0$$

Now add the two series together

$$\frac{dy}{dx}-2xy=1\cdot c_1 x^0+\sum_{n=2}^{\infty} nc_n x^{n-1}-\sum_{n=0}^{\infty} 2c_n x^{n+1} = 0$$

Let k = n - 1 and k = n +1 in each respective sum

$$= c_1+\sum_{k=1}^{\infty} n(k+1)c_{k+1}x^k-\sum_{k=1}^{\infty} 2c_{k-1}x^k$$

$$= c_1 + \sum_{k=1}^{\infty} \left[n(k+1)c_{k+1} - 2c_{k+1} - 2c_{k-1} \right] x^k = 0$$

Therefore $c_1 = 0$ and $(k+1)c_{k+1} - 2c_{k-1} = 0$, k= 1,2,3,…

This is a recurrence relation

$$c_{k+1} = \frac{2c_{k-1}}{k+1}$$

$$k = 1, c_2 = \frac{2}{2} c_0$$

$$k = 2, c_3 = \frac{2}{3} c_1 = 0$$

$$k = 3, c_4 = \frac{2}{4} c_2 = \frac{1}{2} c_0 = \frac{1}{2!} c_0$$

$$k = 4, c_5 = \frac{2}{5} c_3 = 0$$

$$k = 4, c_6 = \frac{2}{6} c_4 = \frac{1}{3 \cdot 2} c_0 = \frac{1}{3!} c_0$$

From this we find

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$= c_0 + c_0 x^2 + \frac{1}{2!} c_0 x^4 + \frac{1}{3!} c_0 x^6 + \dots$$

$$= c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

Ex: Find solutions of $4y'' + y' = 0$ in the form of a power series in x

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<u>Solution about Ordinary Points Power Series Solutions</u>

Suppose the linear second order DE

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = 0$$

is put in standard form

$$y'' + P(x) y' + Q(x) y = 0$$

Definition: Ordinary and Singular Points
A point x_0 is said to be an **ordinary point** of the second order DE if both P(x) and Q(x) in standard form are analytic (*can be written as a power series in x – x₀ with a positive radius of convergence*) at x_0. A point that is not an ordinary point is said to be a **singular point** of each equation.

Every finite value of x is an ordinary point of the differential equation $y'' + (e^x)y' + (\sin x)y = 0$. In particular, **x = 0** is an ordinary point because the two functions are analytic at this point. While $y'' + (e^x)y' + (\ln x)y = 0$ has a singular point at **x = 0** because **ln x** is discontinuous here.

Polynomial Coefficients

We will mainly be interested in polynomial coefficients because they are analytic at every point and rational functions because they are analytic at points where the denominator is not zero. So if $a_2(x)$, a_1(x) and a_0(x) are polynomial with no common factors then P(x) and Q(x) are analytic except where a_2(x)≠0 and singular when a_2(x)=0.

Ex: $(x^2-1)y'' + 2xy' + 6y = 0$

Ex: $ax^2y'' + bxy' + cy = 0$

Ex: $(x^2+1)y'' + xy' - y = 0$

Existence of Power Series Solutions
If $x = x_0$ is an ordinary point of the second order DE, we can always find two linear independent solutions in the form of a power series centered at x_0:

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

A series solution converges at least for $|x - x_0|$ < R, where R is the distance from x_0 to the closest singular point (real or complex).

A solution of the form $y = \sum_{n=0}^{\infty} c_n (x-a)^n$ is said to be a **solution about the ordinary point** x_0. The distance R in the theorem is the minimum value or the lower bound for the radius of convergence of the series solutions of the DE about x_0 .

Finding a Power Series Solution

Finding a power series solution is very similar to finding a particular solution of a nonhomogeneous equation by the method of undetermined coefficients. We substitute $y = \sum_{n=0}^{\infty} c_n x^n$ into the DE, combine series, and then equate all coefficients to the right-hand side of the equation to determine the coefficients c_n. Because the right-

hand side is zero, by the identity property, all coefficients of x must be equal to zero, this does not mean all coefficients are zero. The general solution of the DE is
$y = c_1 y_1(x) + c_2 y_2(x)$ where $c_1 = c_0$ and $c_2 = c_1$

Ex: Solve $y'' + xy = 0$

Since there are no singular points the previous theorem guarantees two power series solutions centered at 0, convergent for |x |< ∞. Substituting

$$y = \sum_{n=0}^{\infty} c_n x^n \;,\; y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \;,\; y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Add the two together letting k = n - 2 in the first series and
k = n +1 in the second series

$$y'' + xy = 2c_2 + \sum_{k=1}^{\infty} \left[(k+1)(k+2)c_{k+2} + c_{k-1}\right] x^k = 0$$

Therefore

$$(k+1)(k+2)c_{k+2} + c_{k-1} = 0, k = 1,2,3\ldots \text{ and } 2c_2 = 0$$

then $c_2 = 0$ and the previous expression is called a recurrence relation, we can solve for c_{k+2} in terms c_{k-1} :

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}, k = 1,2,3\ldots$$

This generates consecutive coefficients of the assumed solution one at a time as we let k take on the consecutive integers:

$$k = 1, c_3 = -\frac{c_0}{2 \cdot 3}$$

$$k = 1, c_4 = -\frac{c_1}{3 \cdot 4}$$

$$k = 1, c_5 = -\frac{c_2}{4 \cdot 5} = 0$$

$$k = 1, c_6 = -\frac{c_3}{5 \cdot 6} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} c_0$$

$$k = 1, c_7 = -\frac{c_4}{6 \cdot 7} = \frac{1}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} c_1$$

$$k = 1, c_8 = -\frac{c_5}{7 \cdot 8} = 0$$

And so on. Now substituting the coefficients just obtained into the original assumption

$$y=\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \ldots$$

We get

$$y = c_0 + c_1 x + 0 - \frac{c_0}{2\cdot 3}x^3 - \frac{c_1}{3\cdot 4}x^4 + 0 + \frac{c_0}{2\cdot 3\cdot 5\cdot 6}x^6 + \ldots$$

By grouping the terms containing co and c1 we obtain

$$y = c_0 y_1(x) + c_1 y_2(x)$$

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 8\cdot 9}x^9 + \ldots = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2\cdot 3\cdots 4\cdot 5\cdot 6}x^{3k}$$

$$y_2(x) = 1 - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + \ldots = x + \sum_{k=1}^{\infty} \frac{(-1)^k}{3\cdot 4\cdots (3k)(3k+1)}x^{3k+1}$$

Ex: Solve $(x^2+1)y'' + xy' - y = 0$

Ex: Solve $y'' - (1+x)y = 0$

Solutions about Singular Points

For the DE $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ we saw that if $x = x_0$ is an ordinary point we can find a solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

What if $x = x_0$ is a singular point? It turns out we may be able to find a solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where r is a constant that must be determined.

Regular and Irregular Singular Points

Working with standard form $y'' + P(x)y' + Q(x)y = 0$ A singular point x_0 is said to be a **regular singular point** of a second order DE if the functions $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ are both analytic at x_0. A singular point that is not regular is said to be an **irregular singular point** of the equation.

For polynomial coefficients of $a_2(x)y''+a_1(x)y'+a_0(x)y=0$ where a_2, a_1, a_0 have no common factors, let $a_2(x)=0$. Form P(x) and Q(x) by reducing $\frac{a_1}{a_2}$ and $\frac{a_0}{a_2}$ to lowest terms, respectively. If the factor $(x-x_0)$ appears at most to the first power in the denominator of P(x) and at most the second power in the denominator of Q(x), then $x=x_0$ is a regular singular point.

Ex: Find the singular points and determine whether regular or irregular for

a. $(x^2-4)y''+3(x-2)y'+5y=0$

b. $x^2(x+1)^2y''+(x^2-1)y'+2y=0$

c. $(1-x^2)y''-2xy'+30y=0$

<u>Frobenius' Theorem</u>

If $x=x_0$ is a regular singular point of the DE, then there exists at least one solution of the form

$$y=(x-x_0)^r\sum_{n=0}^{\infty}c_n(x-x_0)^n=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$$

where the number r is a constant to be determined. The series will converge at least on some interval $0<x-x_0<R$

The trick to this is that we will need to find r. If r is found to be a number that is not a non-negative integer then the corresponding solution is not a power series.

Ex: Because x = 0 is a regular singular point of the differential equation $3xy''+y'=0$ we try to find a solution of the form

$y=\sum_{n=0}^{\infty}c_nx^{n+r-1}$ with $y'=\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}$ and $y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}$

<u>Indicial Equation</u>

r(3r-2) is called the **indicial equation** of the problem and the r values r = 2/3 and r = 0 are called the **indicial roots,** or **exponents,** of the singularity x = 0. In general after substituting and simplifying, the indicial equation is a quadratic in r that results from equating the total coefficient of the lowest power of x to zero. We solve for the two values of r and substitute them into the recurrence relation. Furbenius's Theorem guarantees that at least one solution of the assumed series form can be found.

Ex: Solve $2xy''+(1+x)y'+y=0$

Ex: Solve $xy''+3y'-y=0$