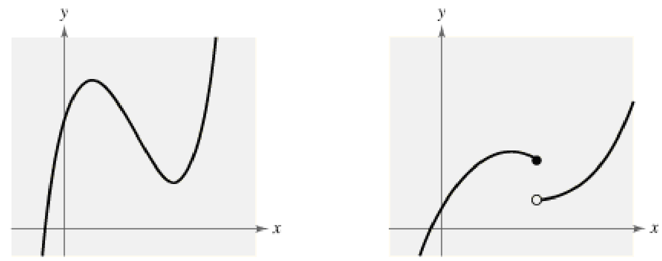
Polynomial Functions

Section Objectives: Students will know how to sketch and analyze graphs of polynomial functions and solve polynomial inequalities.

Characteristics of graphs of polynomial functions

1. Polynomial functions are continuous. This means that the graphs of polynomial functions have no breaks, holes, or gaps.

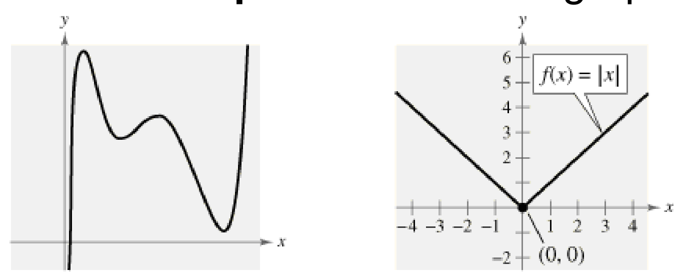


(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

FIGURE 10

2. The **graphs** of polynomial functions **have only nice, smooth turns and bends**. There are **no sharp turns** as in the graph of $y = |x|$.

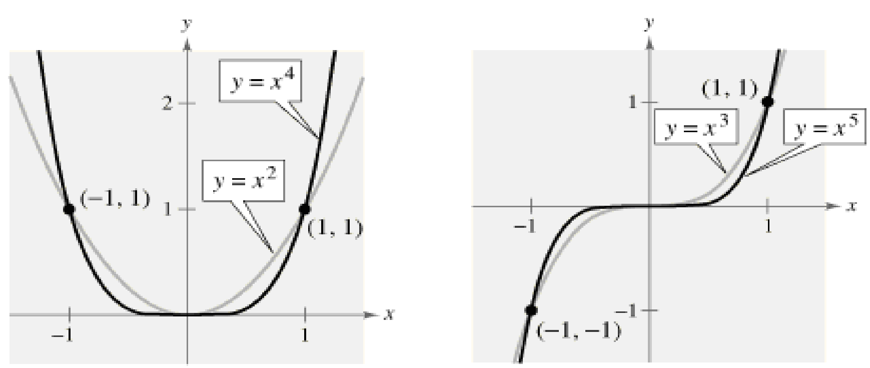


Polynomial functions have graphs with rounded turns.
FIGURE 11

Functions whose graphs have sharp turns are not polynomial functions.
FIGURE 12

You first look at the simplest polynomials, ***f(x) = x^n***. These are called **power functions.** We can break these into two cases, if ***n* is even and *n* is odd**.

Look at two cases:



(a) If n is even, the graph of $y = x^n$ touches the axis at the x-intercept.

(b) If n is odd, the graph of $y = x^n$ crosses the axis at the x-intercept.

FIGURE 13

If *n* is even. It looks like the **Squaring Function**. Note how the graph flattens at the origin as *n* increases.
If *n* is odd. It looks like the **Cubic Function**. Note how the graph flattens at the origin as *n* increases.

Transformations:

Let *c and d* be a positive real numbers. The following changes in the function *y* = *f*(*x*) will produce the stated transformations in the graph of *y* = *f*(*x*).

1. $h(x) = -f(x)$ **reflection** in the ***x*-axis**
2. $h(x) = f(-x)$ **reflection** in the ***y*-axis**
3. $h(x) = a^*f(x)$ $|a| > 1$ **vertical stretch** and $|a| < 1$ **vertical shrink**
4. $h(x) = f(x - c)$ **Horizontal shift** *c* units to the **right**
5. $h(x) = f(x + c)$ **Horizontal shift** *c* units to the **left**
6. $h(x) = f(x) - d$ **Vertical shift** d units **downward**
7. $h(x) = f(x) + d$ **Vertical shift** d units **upward**

Ex: Sketch the graph of the following transformations of power functions.
a) $f(x) = -x^7$
b) $f(x) = -(x + 2)^4$
c) $f(x) = (x - 3)^5 + 4$

Zeros of Polynomial Functions

Zeros (or roots) of a function are all **x values** such that ***f(x) = 0***
In a polynomial function ***f(x)*** of degree ***n*** the following statements are true:

1. The graph of ***f(x)*** has at most ***(n - 1)* turning points** (***turning points*** *are all points where the graph changes from increasing to decreasing or vice versa*).
2. The graph of ***f(x)*** has at most ***n*** real zero's (x-intercepts).

Finding the zeros of a function is one of the most important problems in algebra.

Real zeros of a polynomial

The following are equivalent statements, where ***f(x)*** is a polynomial function and ***a*** is a real number.

1*. a* is a **zero** of ***f(x)***.
2. *x = a* is a **solution** of the equation **f(x) = 0**.
3. *(x – a)* is a **factor** of **f(x)**.
4. *(a, 0)* is an **x-intercept** of the graph of ***f(x)***.

Ex: Find the Real Zero's (x-intercepts) of the graph of

$$f(x) = x^3 - x^2 - x + 1.$$

Ex: Find x-intercepts of the graph of

$$f(x) = -2x^4 + 2x^2$$

Find the possible number of turning points?

The Leading Coefficient Test

As ***x*** **moves** to the **left** or to the **right** without bound the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1 + a_0 x^0$$

eventually rises or falls in the following manner:

$f(x) = a_n x^n + ...$	$a_n > 0$	$a_n < 0$
n even	$\nwarrow \quad \nearrow$	$\swarrow \quad \searrow$
n odd	$\swarrow \quad \nearrow$	$\nwarrow \quad \searrow$

*Note: This **test only determines the behavior at the endpoints!!!!!!!!***

Ex: Look at the right-hand and left-hand behavior of the graph of each function.

a) $f(x) = -x^4 + 7x^3 - 14x - 9$

b) $g(x) = 5x^5 + 2x^3 - 14x^2 + 6$

Steps to Graph a polynomial Function

1. Apply the Leading Coefficient Test *(LCT)*
2. Locate the Intercepts, (x - int: f(x) = 0 and y-int: f(0)=)
3. Plot a Few Check Points
4. Graph the Function

Ex: Sketch the graph of $f(x) = x^3 - 2x^2$

Long Division of Polynomials

We use long division to find factors of a polynomial.

For $f(x) = 6x^3 - 19x^2 + 16x - 4$ if you look at the graph one **x-intercept** is **x = 2** therefore **x = 2 is a zero** and **(x - 2) is a factor** of ***f(x)***. This means there is some **2nd degree** polynomial **q(x)** such that

$$f(x) = (x-2) \bullet q(x)$$

We can use long division to find q(x).

The Division Algorithm.

For all polynomials ***f***(***x***) and ***d***(***x***) such that ***d***(***x***) $\neq$ **0** and the **degree of *d*** is **less than or equal to** the **degree of *f***, there exist unique polynomials ***q***(***x***) and ***r***(***x***) such that :

$$\underbrace{\frac{f(x)}{d(x)}}_{\text{(Improper)}} = q(x) + \underbrace{\frac{r(x)}{d(x)}}_{\text{(Proper)}}$$

where *r*(*x*) = 0 or the degree of *r* is less than the degree of *d*.

If the remainder **r(x) is zero**, d(x) divides evenly into f(x)
We can also write using the division algorithm as:

$$f(x) = d(x)q(x) + r(x)$$

divided = divisor • quotient + remainder

Before you apply Division Algorithm

1. Write the dividend and divisor in descending powers of the variable
2. Insert placeholders with zero coefficients for missing powers.

Ex: Divide $x^2 - 3x + 5$ by $x + 1$.

***Tip**: Remember to subtract every term*

Ex: Divide $x^3 - 1$ by $x - 1$.

Synthetic Division *(Short cut for long division)*

Divide $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x^1 + a_0 x^0$ by $x - k$

1. List the coefficients of the dividend inside the bracket
2. Place k outside the bracket *(watch your signs)*
3. Vertical pattern: Add terms
4. Diagonal pattern: Multiply by k
5. Result gives you the coefficients of the quotient and the remainder which is a constant

The previous procedure applies only when the divisor is of the form **x – k**, and when every descending power of x has a place in the dividend.

Ex: Use synthetic division to divide $\mathbf{4x^4 - 2x^2 - x + 1}$ by **x + 2.**

The Remainder Theorem:

If a polynomial **f(x)** is divided by **x – k**, the remainder **r = f(k)**.

Basically this says that synthetic division can be used to evaluate a function at **x = k**

Ex: Use the Remainder Theorem to find **f (-2)** for

$$f(x) = 4x^3 - 2x^2 + x + 3$$

The Factor Theorem:

A polynomial **f(x)** has a **factor x – k**, **if and only if f(k) = 0**.

This along with the remainder theorem tells you that x – k is a factor of f(x) if the remainder is zero.

Ex: Show that **x – 1** is a factor of $\mathbf{f(x) = x^4 - 1}$.

Ex: Show that **(x – 2)** and **(x + 3)** are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$$

Uses of the Remainder Theorem and Synthetic Division

The remainder ***r*** obtained in the synthetic division of ***f(x)*** by ***x – k*** provides the following info:

1. The remainder ***r*** gives the value of ***f* (x)** at ***x = k*** that is ***r = f(k).***
2. If ***r* = 0**, **(*x – k*)** is a **factor** of ***f(x)*** .
3. If ***r* = 0**, **(*k* , 0)** is an ***x* - intercept** of the graph of ***f(x)***

A **nth degree** polynomial can have **at most n real zero's**

Some zeros may also be complex these do not relate to x intercepts

Repeated Zero's

Note that in the previous example, **1** is a **repeated zero**.

In general, a factor $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity k**.

1. If **k is odd**, the graph **crosses** the **x-axis** at $x = a$.
2. If **k is even**, the graph **only touches** the **x-axis** at $x = a$.

A **polynomial function** can **only change signs** at its **zero's**. **Between two** consecutive **zeros** a **polynomial** is **either entirely pos. or neg**. This relates to the graph. These x-intercepts divide the x axis into intervals we will call **Test Intervals**

Ex: Find all the real zeros along with their multiplicities for

$$f(x) = x^4 - 2x^3 - 3x^2 + 8x - 4$$

The Rational Zero Test

Relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of a polynomial

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 + a_0$ has integer coefficients with $a_n \neq 0$ and $a_0 \neq 0$, then all possible rational zeros of $f(x)$ will be of the form

Rational Zero = p/q

p is a factor of a_0 (constant)
q is a factor of a_n (leading coefficient).

To use: *List all possible rational #'s whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient. Next use trial and error method and synthetic division to determine which are actual zeros.*

Ex: Find the rational zeros of $f(x) = x^3 - 7x - 6$

Ex: Find rational zeros of $f(x) = x^3 + x + 1$

Polynomial Inequalities

Location Theorem: Suppose that a function f is continuous on an interval I that contains numbers a and b. If $f(a)$ and $f(b)$ have opposite signs, then the graph of f has at least one zero between a and b.

If we find where a polynomial function is equal to zero (*i.e. find the zeros*) then everywhere else this function will have to be positive or negative. Then all we have to do is choose the answers that satisfy the inequality.

Steps for solving a Polynomial Inequality:

1. Get a zero on one side of the inequality
2. If possible, factor the polynomial
3. Determine where the polynomial is zero (*i.e. find the zeros*)
4. Graph the points where the polynomial is zero *(the points from the previous step)* on a number line and pick a **test point** from each of the intervals to determine its sign
5. Determine which intervals satisfy the inequality, write your answer in interval notation.

Ex: Solve $x^3 - 2x^2 - 5x + 6 > 0$

Ex: Solve $2x^3 - 15x \le -7x^2$