Plane Curves, Parametric Equations, and Polar Coordinates:

Until now we have been representing a graph by a single equation involving two variables. Here we begin to study situations in which three variables are used to represent a curve in the rectangular coordinate plane.

Suppose an object is propelled into the air at an angle of $45°$. The path can be given by a function, $y = f(x)$. This equation does not give you the entire story. It tells you where a given object is and the path it traveled, but it does not tell you when the object was at a given point.

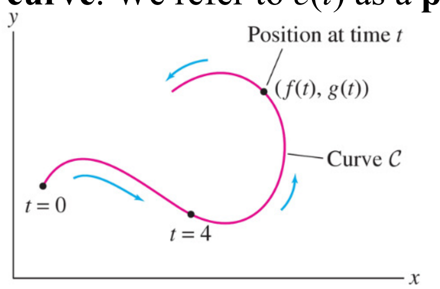
If a particle is moving along a curve *C* in the plane, we can describe the particle's motion by specifying its coordinates as function of time *t*:

$$x = f(t), \quad y = g(t).$$

In other words, at time *t*, the particle is located at point

$$c(t) = (f(t), g(t)).$$

The first set of equations are called the **parametric equations**, and *C* is called the **parametric curve**, or **plane curve**. We refer to *c*(*t*) as a **parametrization** with parameter *t*.



1 Particle moving along a curve $\mathcal{C}$ in the plane.

Since *x* and *y* are functions of *t*, we often write $c(t) = (f(t), g(t))$ instead of $(f(t), g(t))$.

Curve Sketching:

To sketch a parametric curve by hand plot the points (*x,y*) determined from a value chosen for the parameter *t*. By plotting the resulting points in order of increasing value of *t* the curve is traced out in a specific direction. This is called the **orientation of the curve**.

Ex: Sketch the curve with parametric equations

$$x = 2t - 4 \text{ and } y = 3 + t^2 .$$

Eliminating the Parameter:
A parametric curve $c(t)$ need not be the graph of a function. If it is, however, it may be possible to find the function $f(x)$ by "eliminating the parameter".

Ex: Describe the curve of the previous example $c(t) = (2t - 4, 3 + t^2)$ in the form of $y = f(x)$.

Common Parametrizations:
Two of the most common parametrizations that we common across are lines and circles.

Parametrization of a Line
The line through $P = (a,b)$ of slope m is parametrized by

$$x = a + rt, \quad y = b + st \qquad -\infty < t < \infty$$

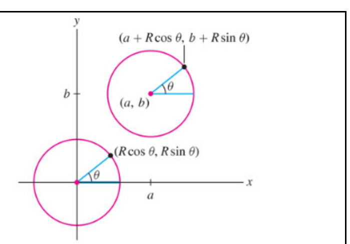
For any r and s such that $m = s/r$ and $r \neq 0$.

Ex: Find the parametric equations for the line through $P = (3,-1)$ with slope $m = 4$.

Ex: Find the parametric equations for the line through the points $(1,2)$ and $(-5,3)$.

Parametrization of a Circle:
The circle of radius R and center (a,b) oriented counterclockwise is parametrized by

$$x = a + R\cos\theta, \quad y = b + R\sin\theta$$



Parametrization of a circle of radius R with center (a, b).

Where θ varies over a half open interval of length 2π such as $[0,2\pi)$.

For an ellipse centered at (a,b) the parametrization would be

$$x = a + R\cos\theta\,, \; y = b + S\sin\theta$$

where $S \neq R$.

Ex: Sketch the curve represented by $x = 3\cos\theta$ and $y = 4\cos\theta,\quad 0 \le x \le 2\pi$ then eliminate the parameter and find the corresponding rectangular equation.

Some interesting parametrizations:

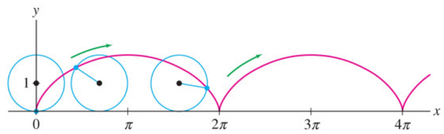
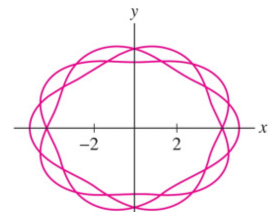


FIGURE 10 A cycloid.

$x(t) = t - \sin t,\ y(t) = 1 - \cos t$



The parametric curve $x = 5\cos(3t)\cos\left(\frac{2}{3}\sin(5t)\right)$, $y = 4\sin(3t)\cos\left(\frac{2}{3}\sin(5t)\right)$.

Parametric Equations and Calculus

Parametric Form of a Derivative:
If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x,y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{dx}{dt} \neq 0$$

Ex: Find the equation of the tangent line for the curve given by $x = \sin t$ and $y = \cos t$ when $t = \pi$.

Polar Coordinates

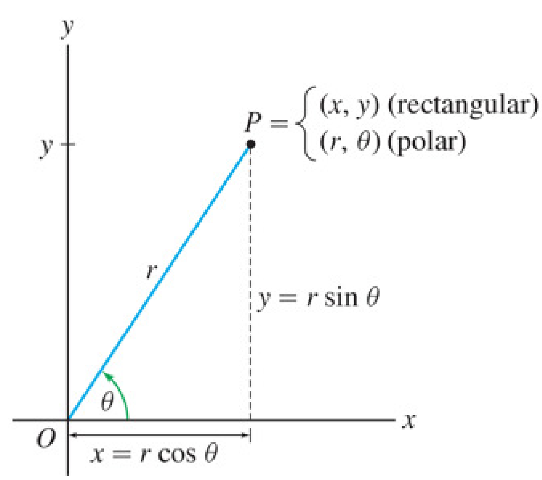
Polar coordinates (r, θ) of a point (x, y) in the Cartesian plane are another way to plot a graph. The r represents the distance you move away from the origin and θ represents an angle in standard position.

To convert from rectangular to polar

$$r^2 = x^2 + y^2$$
$$\theta = \arctan(y/x)$$

To convert from polar to rectangular

$$x = r\cos\theta$$
$$y = r\sin\theta$$



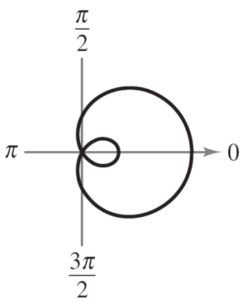
Ex: Convert (3, -3) to polar coordinates

Ex: Convert $\left(2, \frac{2\pi}{3}\right)$ to rectangular coordinates

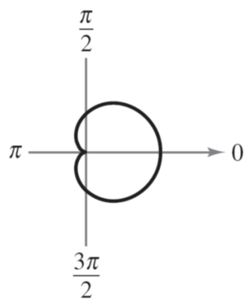
Ex: Convert the rectangular equation $x^2 + y^2 = 1$ to polar coordinates.

Ex: Find the polar equation of the line through the origin with slope $m = 3/2$.

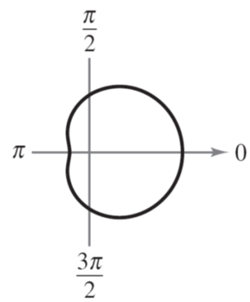
<u>Some special polar figures:</u>



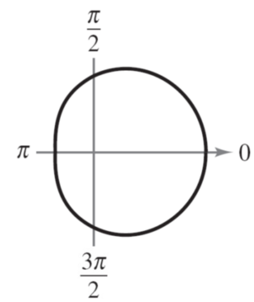
$\frac{a}{b} < 1$
Limaçon with inner loop



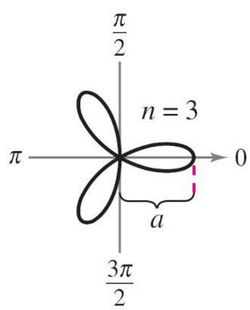
$\frac{a}{b} = 1$
Cardioid (heart-shaped)



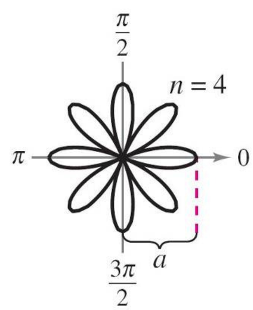
$1 < \frac{a}{b} < 2$
Dimpled limaçon



$\frac{a}{b} \geq 2$
Convex limaçon



$r = a \cos n\theta$
Rose curve



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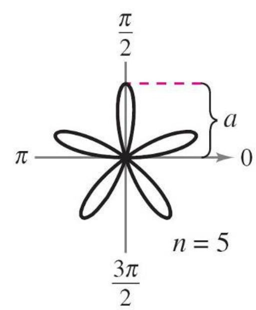
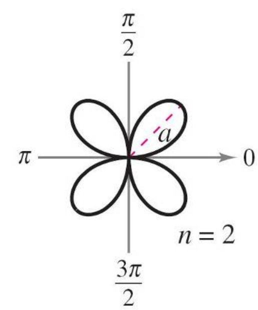


$r = a \sin n\theta$
Rose curve



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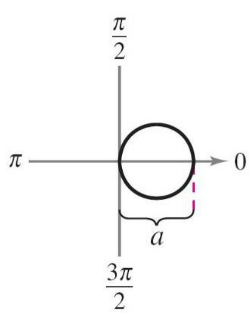
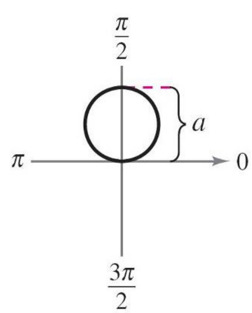
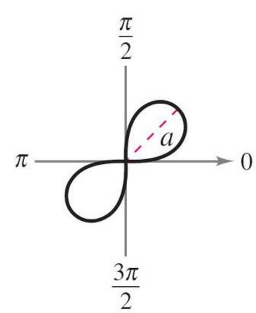


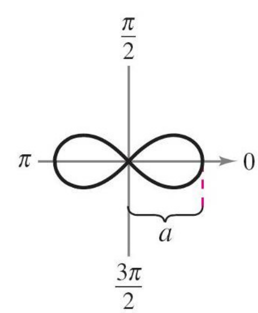
$r = a \cos \theta$
Circle



$r = a \sin \theta$
Circle



$r^2 = a^2 \sin 2\theta$
Lemniscate



$r^2 = a^2 \cos 2\theta$
Lemniscate