Multiple Integration

Double Integrals, Volume, and Iterated Integrals

In single variable calculus we looked to find the area under a curve f(x) bounded by the x-axis over some interval using summations then that led to using integrals. That same process can be translated over to multi-variable calculus and volume. Instead of calculating the areas of a rectangles we will be calculating the volume of rectangular prisms.

Single Variable : $\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)dx$ where $\Delta x = \frac{b-a}{n}$

Now we are looking to find a volume of solid S that lies below a surface z = f(x,y) and above some rectangle R in the x – y plane where $R = [a,b]*[c,d]$, [a,b] is the interval over the x – axis and [c,d] is the interval over the y – axis.

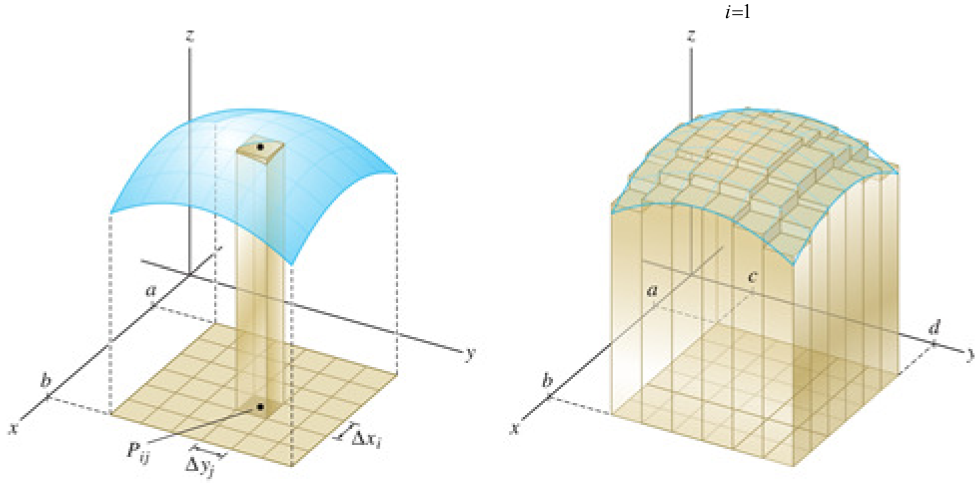
We begin by sub-diving R into sub-rectangles. So [a,b] is divided into sub-intervals $\Delta x = \frac{b-a}{n}$ and [c,d] as well $\Delta y = \frac{d-c}{n}$. Every sub-rectangle should have area:

$$\Delta A = \Delta x \Delta y$$

Then the height of each rectangular prism is z =f(x_i,y_i). So to find the volume of each rectangular prism we use

$$V = f(x_i, y_i)\Delta A$$

Now if we want to add all these prisms up we use the sum $\sum_{i=1}^{n} f(x_i, y_i)\Delta A$



If we think about the Riemann sum from calculus I, we can take the limit of this sum as n→∞ (n = the number of sub-rectangles) or let the diagonal of each sub-rectangle go to zero) and this will give the actual volume and can be found using an integral (or two).

Double Integrals:

If f is defined on a closed, bounded region R in the xy-plane, then the double integral of f over R is given by

$$\iint_R f(x,y)\,dA = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i,y_i)\Delta A$$

provided the limit exists.

If z =f(x,y) ≥ 0 for all (x,y) in R, then the above double integral will represent the volume of the solid that lies above R and below f.

All the usual properties of single integrals apply to double integrals

In order to evaluate the double integral we can express them as iterated integrals

Iterated Integrals:

To apply the Fundamental Theorem of Calculus to double integrals we use the following methods called iterated integrals

$$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy = \int_a^b\int_c^d f(x,y)\,dydx = \int_c^d\int_a^b f(x,y)\,dxdy$$

This process can be thought of as integrating one variable at a time. Remember not every iterated integral has to represent area.

Fubini's Theorem:

The double integral of a continuous function f(x,y) over a rectangle R = [a,b]*[c,d] is equal to the iterated integral

$$\iint_R f(x,y)\,dA = \int_a^b\int_c^d f(x,y)\,dydx = \int_c^d\int_a^b f(x,y)\,dxdy$$

Ex: Evaluate $\iint_R (x^2 - y^2)\,dA$ if $R = \{-1 \le x \le 2, -2 \le y \le 1\}$

Ex: Evaluate $\iint_R (2x + 3y^2)\,dA$ if $R = \{-2 \le x \le 3, 1 \le y \le 4\}$

Ex: Find the volume between the graph of $f(x,y) = 16 - x^2 - 3y^2$ and the rectangle $R = [0,3]*[0,1]$

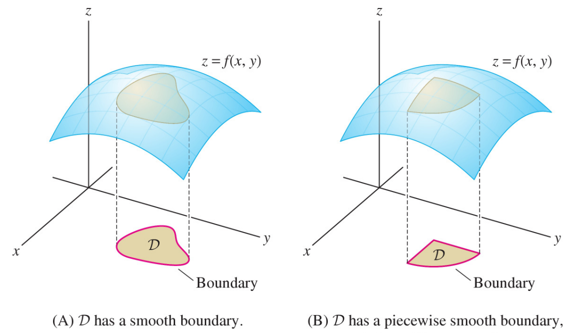
When a function is a product $f(x,y) = g(x)h(y)$ the double integral over a rectangle [a,b]*[c,d] is simply the product of the single integrals.

$$\iint_R f(x,y)\,dA = \iint_R g(x)h(y)\,dxdy = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

Ex: Evaluate $\iint_R \frac{y}{x+1}\,dA$ if $R = \{0 \le x \le 2, 0 \le y \le 4\}$

Double Integrals over More General Regions:

What if the region R under the surface is not a rectangle but instead a more general region? Can we still find its volume? -----YES!!!
(this is more common)



(A) D has a smooth boundary. (B) D has a piecewise smooth boundary, consisting of three smooth curves joined at the corners.

FIGURE 1

We redefine the function to be itself in the region and zero outside the region by the following

$$F(x,y)=\begin{cases} f(x,y) \text{ if } (x,y)\in D \\ 0 \text{ if } (x,y)\in R \text{ but } (x,y)\notin D \end{cases}$$

where D is the domain of the region

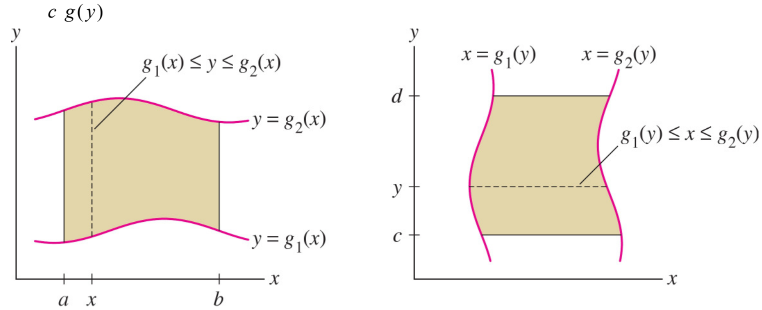
In order for this redefining to work and actually find the volume, one set of limits of integration must be constant. We set it up so that the inside limits can be functions of the opposite variable and the outside limits are numerical.

i. Given $\iint_R f(x,y)dA$ such that $R=\{(x,y)|a\le x\le b, g(x)\le y\le h(x)\}$ then

$$\iint_R f(x,y)dA=\int_a^b\int_{g(x)}^{h(x)} f(x,y)dydx$$ if the region is vertically simple

ii. Given $\iint_R f(x,y)dA$ such that $R=\{(x,y)|g(y)\le x\le h(y), c\le y\le d\}$ then

$$\iint_R f(x,y)dA=\int_c^d\int_{g(y)}^{h(y)} f(x,y)dxdy$$ if the region is horizontally simple



(A) Vertically simple region (B) Horizontally simple region

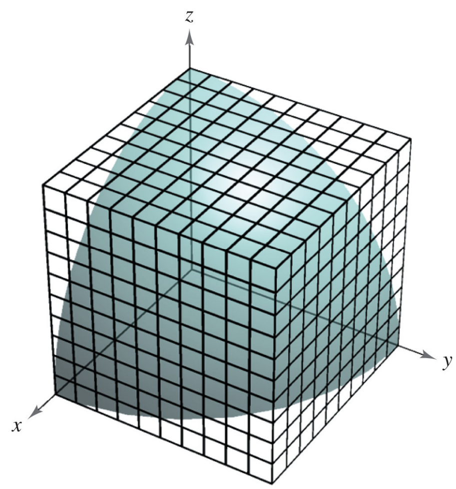
FIGURE 6

Ex: Evaluate $\int_1^4\int_1^{\sqrt{x}} 2ye^{-x}dydx$ *(see what happens if you switch the order of integration)*

Ex: Find the volume of the solid region bounded by the paraboloid $z=4-x^2-2y^2$ and the xy-plane.

Triple Integrals

We define single integrals for functions of single variables and double integrals for functions of two variables. Now we define triple integrals for functions of three variables.



Consider the a function $w = f(x, y, z)$ that is continuous on a rectangular box, $B = \{(x, y, z) | a \le x \le b, c \le y \le d, e \le z \le f\}$.

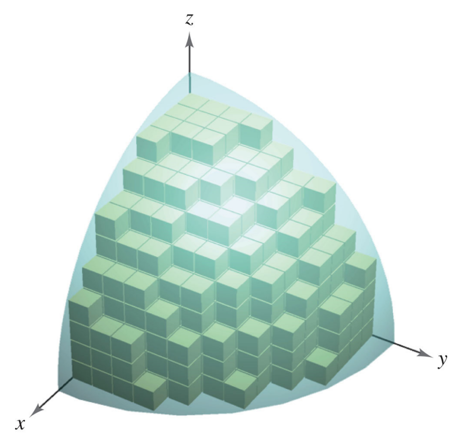
Then, encompass B with a network of boxes and form the inner partition consisting of all boxes lying entirely within B. The volume of each sub-box is

$$\Delta V = \Delta x \Delta y \Delta z$$

If we start stacking the boxes up to the function we can use the Reimann sum to approximate the volume by adding up the sub-boxes where (x_i, y_i, z_i) is a sample point in each box

$$\sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta V$$

Then by letting the number of boxes go to infinity (or the diagonal of each box go to zero) we can evaluate the volume.



Triple Integrals:

If f is continuous over a bounded solid region B, then the triple integral of f over B is define by

$$\iiint_B f(x, y, z) dV = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta V$$

provided the limit exists.

Fubini's Theorem for Triple Integrals:

The triple integral of a continuous function f(x, y, z) over a box B is equal to the iterated integral

$$\iiint_B f(x, y, z) dV = \int_a^b \int_c^d \int_e^f f(x, y, z) dz dy dx$$

this iterated integral can be evaluated in any order.

Ex: Evaluate $\iiint_B (xyz^2) dV$ where $B = \{(x, y, z) | 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$

What do we do when the region is more generalized then a box? (Meaning the limits of integration are functions instead of numbers)

More General Regions for Triple Integrals in 3-D Space:

Let f be continuous on a solid region B defined by

$a \le x \le b$, $h_1(x) \le y \le h_2(x)$, $g_1(x,y) \le z \le g_2(x,y)$ *(simple with respect to the order dzdydx)*

Where h_1, h_2, g_1, and g_2 are continuous functions. Then,

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dzdydx$$

If this region is simple in different orders it would work out in a similar pattern accordingly.

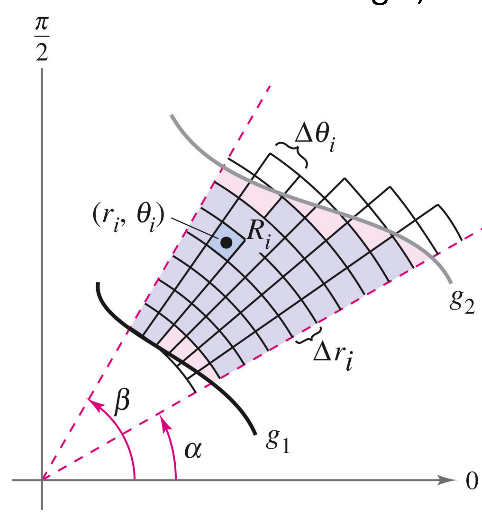
Ex: Evaluate $\int_0^9 \int_0^{y/3} \int_0^{\sqrt{y^2-9x^2}} z\,dzdxdy$

Ex: Integrate $f(x,y,z) = z$ over the region lying below the upper hemisphere of radius 3 and the above the triangle in the xy-plane bounded by the line x = 1, y = 0 and x = y.

Ex: Find the volume of the solid in the octant x, y, z ≥ 0 bounded by the planes $x + y + z = 1$ and $x + y + 2z = 1$

Integration in Polar, Cylindrical, and Spherical Coordinates

If f(x,y) is a continuous function and you need to integrate it over a circle at the origin, then is much easier to use polar coordinates.



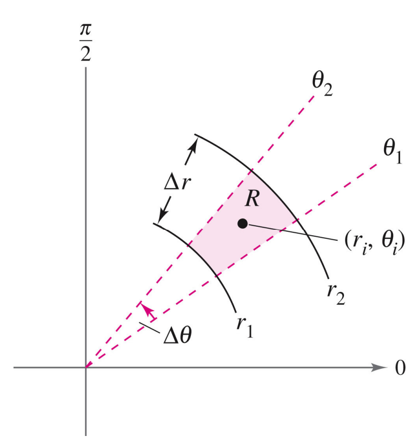
Recall: To convert from polar to rectangular:

x = rcosθ ; y = rsinθ

To convert from rectangular to polar:

$r^2 = x^2 + y^2$; Θ = arctan(y/x)

To define a double integral of a continuous function $z = f(x,y)$ in polar coordinates, consider a region R bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$. Instead of partitioning R into small rectangles, use a partition of small **polar sectors**. On R superimpose a polar grid made of rays and circular arcs. The area of a specific polar sectors lying entirely within R is



$$\Delta A = r\Delta r\Delta\theta, \text{ where } \nabla r = r_2 - r_1 \text{ and } \Delta\theta = \theta_2 - \theta_1$$

This implies that the volume of a solid with height

$$f(r\cos\theta, r\sin\theta)r\Delta r\Delta\theta$$

and we have

$$\iint_R f(r\cos\theta, r\sin\theta)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

where $g_1(\theta) \le r \le g_2(\theta)$ and $\alpha \le \theta \le \beta$

Ex: Let R be the region that lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$. Evaluate $\iint_R (x^2 + y)dA$.

Ex: Let R be the region that lies between the y ≥ 0 and $x^2 + y^2 \le 1$. Evaluate $\iint_R (y(x^2 + y^2)^3)dA$ using polar.

Ex: Find the volume of the solid that lies above the circle $x^2 + y^2 \le 4$ in the xy-plane and below the hemisphere $z = \sqrt{16 - x^2 - y^2}$

<u>Cylindrical Coordinates:</u>

Cylindrical coordinates are useful when the domain has axial symmetry, symmetry with respect to an axis. Usually when we have the expression $x^2 + y^2$.

Recall: x = rcosθ

y = rsinθ

z = z

To set up the triple integral in cylindrical coordinates we assume that the domain of integration W can be described as the region between two surfaces $z_1(r,\theta) \le \theta \le z_2(r,\theta)$ lying over a domain D in the xy-plane with polar description $D: \theta_1 \le \theta \le \theta_2,\ r_1(\theta) \le \theta \le r_2(\theta)$

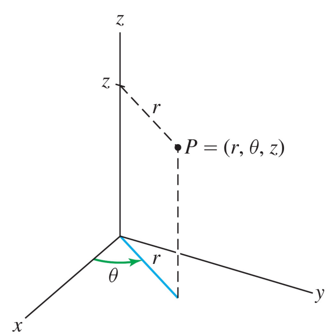


FIGURE 8 Cylindrical coordinates.

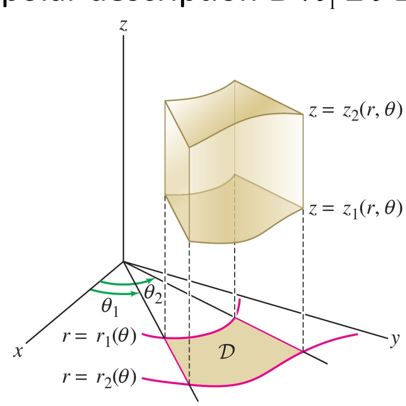


FIGURE 9 Region described in cylindrical coordinates.

A triple integral over W can be written as

$$\iiint_W f(x,y,z)dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\underline{rdzdrd\theta}$$

Ex: Evaluate by converting to cylindrical coordinates: $\iiint_W (x^2 + y^2)dV$ where

$W = \left\{(x,y,z) \middle| -2 \le x \le 2, -\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}, \sqrt{x^2 + y^2} \le z \le 2\right\}$.

Spherical Coordinates:

Recall: Rectangular → Spherical : x = ρsinϕcosθ, y = ρsinϕsinθ, z = ρcosϕ

Spherical → Rectangular: $\rho = \sqrt{x^2 + y^2 + z^2}$, tanθ = y/x, cosϕ = z/ρ

As we saw the change of variables formula in cylindrical coordinates can be summarized by $dV = r dr d\theta dz$. With spherical the same things can be done to summarize using

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

By working with a the region known as a **spherical wedge**

$$W = \{(\rho, \phi, \theta \,|\, \theta_1 \le \theta \le \theta_2, \phi_1 \le \phi \le \phi_2, \rho_1 \le \rho \le \rho_2\}.$$

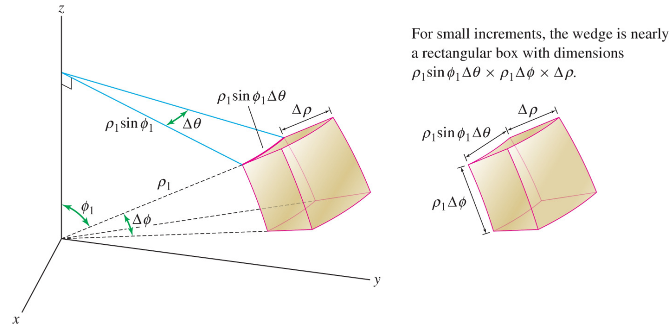


FIGURE 13 Spherical wedge.

The triple integral over w can be written as

$$\iiint_W f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)dV = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1(\theta,\phi)}^{\rho_2(\theta,\phi)} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Ex: Evaluate $\iiint_W e^{(x^2+y^2+z^2)^{3/2}} dV$ where W = sphere with radius 1.

Change of Variables: Jacobians

When we convert from rectangular to another system an extra factor appears for our value of dV (ie. rect.→polar gives an extra factor r)

Where do these extra factors come from?????????

We can transform rectangular coordinates to any system this is logical by changing the variable. When we do this we must also adjust the differentials dA or dV. The adjustment/extra factors come from the change in variables and are called the **Jacobian.**

The Jacobian:

If x = g(u, v) and y = h(u, v), then the **Jacobian** of x and y with respect to u and v, denoted by $\frac{\partial(x,y)}{\partial(u,v)}$, is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

In three independent variable this extends to a 3X3 matrix

Ex: Find the Jacobian for $x = -\frac{1}{2}(u - v), y = \frac{1}{2}(u + v)$

Ex: Find the Jacobian for x = rcosθ , y = rsinθ

Change of Variables for Double Integrals

Let R be a vertically or horizontally simple region in the xy-plane, and let S be a vertically or horizontally simple region in the uv-plane. Let T from S to R be given by T(u, v) = (x, y) =(g(u, v),h(u, v)), where g and h have continuous first partial derivatives. Assume the T is one-to-one except possibly on the boundary of S. If f is continuous on R, and $\partial(x,y)/\partial(u,v)$ is nonzero on S, then

$$\iint_R f(x,y)\,dxdy = \iint_s f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

Ex: Use the transformation $x=\dfrac{u}{v+1}$ and $y=\dfrac{uv}{v+1}$ to compute $\iint_D (x+y)\,dxdy$ where D is the region bounded by y = 2x, y = x, y = 3 – x and y = 6 – x.