Linear Differential Equations of Higher Order

BasicTheory:

Initial–Value Problems

Solve: $a_n(x)\dfrac{d^n y}{dx^n}+a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\dfrac{dy}{dx}+a_0(x)y=g(x)$

Subject to: $y(x_0)=y_0,\ y'(x_0)=y_1,\ \ldots\ ,\ y^{(n-1)}(x_0)=y_{n-1}$

Existence of a Unique Solution:

Let $a_n(x)$, $a_{n-1}(x)$, … ,$a_1(x)$, $a_0(x)$ and $g(x)$ be continuous on an interval *I*, and let $a_n(x) \neq 0$ for every *x* in this interval. If $x = x_0$ is any point in this interval, then a solution *y(x)* of the initial value problem exists on the interval and is unique.

Ex: Find the solution to $y''+y=4x+10\sin x$ subject to $y(\pi)=0,\ y'(\pi)=2$ given that $y=c_1\cos x+c_2\sin x+4x-5x\cos x$ is a general solution to the D.E. Is that solution unique?

BoundaryValueProblems

Another type of problem consists of solving a linear DE of order two or greater in which the dependent variable *y* or its derivatives are specified at different points. A problem such as

Solve: $a_2(x)\dfrac{d^2 y}{dx^2}+a_1(x)\dfrac{dy}{dx}+a_0(x)y=g(x)$

Subject to: $y(x_0)=y_0,\ y(x_1)=y_1$

is called a **two point boundary value problem** or **BVP**. The prescribed values $y(x_0) = y_0$ and $y(x_1) = y_1$ are called boundary conditions. A solution of the foregoing problem is a function satisfying the D.E. on some interval *I*, containing x_0 and x_1, whose graph passes through the two points (x_0,y_0) and (x_1,y_1).

Ex: Solve $y''+16y=0$ subject to $y(0)=0,\ y(\pi/2)=0$, given that $y=c_1\cos 4x+c_2\sin 4x$ is a general solution to the D.E.

Difference between and IVP and BVP:

- In an IVP all values needed to solve a particular problem are specified at a single point (x_0)
- In a BVP all values needed to solve a particular problem are specified at different points (x_0, x_1, etc)

Linear Dependence and Linear Independence

A set of functions $f_1(x), f_2(x),\ldots,f_n(x)$ is said to be **linearly dependent** on an interval *I* if there exists constants $c_1, c_2, \ldots ,c_n$, not all zero, such that

$$c_1 f_1(x)+c_2 f_2(x)+\ \ldots+c_n f_n(x)=0$$

for every x in the interval.

If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

In other words a set of functions is linearly independent on an interval I if and only if the only constants for which

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0$$

for every x in the interval are $c_1 = c_2 = \ldots = c_n = 0$

Ex: Are the functions $f_1(x) = \sin 2x$ and $f_2(x) = \sin x \cos x$ linearly dependent or independent?

Ex: Are the functions $g_1(x) = x^2$ and $g_2(x) = \ln x$ linearly dependent or independent?

Ex: Are the functions $h_1(x) = \sqrt{x} + 5$, $h_2(x) = \sqrt{x} + 5x$, $h_3(x) = x - 1$ and $h_4(x) = x^2$ linearly dependent or independent?

Also note that a set of functions $f_1, f_2, f_3, \ldots, f_n$ is linearly dependent on an interval if at least one function can be expressed as a linear combination of the remaining functions.

Solutions of a D.E.

We are primarily interested in linearly independent solutions of linear D.E. To determine whether a set of solutions of an nth order linear D.E. is linearly independent can be done using determinants.

Wronskian:

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possess at least $n - 1$ derivatives. The determinant of

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = W(f_1, f_2, \ldots, f_n)$$

is called the **Wronskian** of the functions.

Criterion for Linearly Independent Functions:

The set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is **linearly independent** on I if and only if $W(f_1, f_2, \ldots, f_n) \neq 0$ for at least one point in the interval. The converse is also true

Ex: Are the following examples linearly independent?

A. $f_1(x) = \sin^2 x,\ f_2(x) = 1 - \cos 2x$

B. $g_1(x) = e^{m_1 x},\ g_2(x) = e^{m_2 x},\ m_1 \neq m_2$

C. $h_1(x) = e^{\alpha x} \cos \beta x,\ h_2(x) = e^{\alpha x} \sin \beta x,\ \alpha \text{ and } \beta \text{ are real numbers}$

D. $z_1(x) = e^x, z_2(x) = xe^x, z_3(x) = x^2 e^x$

Solutions of linear Differential Equations

A linear nth order DE of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be **homogeneous**, whereas an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

with *g(x)* not identically zero, is said to be **nonhomogeneous**

(The word homogeneous here does not refer to the coefficients that are homogeneous functions)

Ex: $2y'' + 3y' + 5y = x$ is a nonhomogeneous second order linear differential equation and $2y'' + 3y' + 5y = 0$ is the associated homogeneous equation.

For now on we will make the following assumptions when stating definitions and theorems about linear equations on some interval *I*,

1. The coefficient functions $a_i(x)$, $i = 0,1,2,\ldots,n$ are continuous
2. ***g(x)*** is continuous
3. $a_n(x) \neq 0$ for every *x* in the interval.

SuperpositionPrinciple–HomogeneousEquations:

Let $y_1, y_2, \ldots, y_k$ be linearly independent solutions of the homogeneous *n*th order differential equation on an interval *I*, then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \ldots + c_ky_k(x),$$

where the c_i, $i = 0,1,2\ldots,k$ are arbitrary constants, is also solution on the interval, we call this the **general solution** of the homogeneous D.E.

Any set $y_1, y_2, \ldots, y_n$ of linearly independent solutions of the homogeneous linear nth order differential equation on an interval *I* is said to be a **fundamental set of solutions** on the interval.

Corollaries to the Superposition Theorem:

- A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear DE is also a solution.
- A homogeneous linear DE always possesses the trivial solution $y = 0$.

Ex: The functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear equation $x^3y''' - 2xy' + 4y = 0$ on the interval $(0,\infty)$. By the superposition principle the linear combination $y = c_1x^2 + c_2x^2 \ln x$ is also a solution of the equation on the interval.

Nonhomogeneous Linear Differential Equations

Any function y_p, free of arbitrary parameters, that satisfies a nonhomogeneous linear D.E. is said to be a **particular solution** or **particular integral** of the equation. An easy example would be $y_p = 3$ is a particular solution to $y'' + 9y = 27$. The particle solution isn't necessary restricted to constants.

Let y_p be a given (or particular) solution of the nonhomogeneous linear *n*th order differential equation on the interval *I*, and let

$$y_c = c_1 y_1(x) + c_2 y_2(x) + ... + c_k y_k(x)$$

denote the general solution of the associated homogeneous equation on the interval, then the **general solution** of the nonhomogeneous equation on the interval is defined to be

$$y = y_c + y_p = c_1 y_1(x) + c_2 y_2(x) + ... + c_n y_n(x) + y_p(x)$$

Ex: The nonhomogeneous linear differential equation $y''' - 6y'' + 11y' - 6y = 3x$ has a particular solution $y_p = -\frac{11}{12} - \frac{1}{2}x$ and its associated homogeneous equations has a general solution $y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$. Find the general solution to the nonhomogeneous D.E.

Reduction of Order

If you have a known solution to a second order linear differential equation one interesting thing that occurs with these types of equations is that you can use that solution to construct a second solution.

Suppose $y_1(x)$ is a known solution to $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$. We assume just like previously that $a_2(x) \neq 0$ for every *x* in some interval *I*. The process we use to a find a second solution $y_2(x)$ consists of **reducing the order** of the equation to a first order linear D.E., which we already know how to solve, through substitution.

Suppose that $y_1(x)$ is a nontrivial solution of the previous D.E. and that $y_1(x)$ is defined on *I*. We seek a second solution, $y_2(x)$, so that $y_1(x)$, $y_2(x)$, are a linearly independent on *I*. If $y_1(x)$ and $y_2(x)$ are linearly independent then the quotient $y_2(x)/ y_1(x)$, is non-constant that is $y_2(x)/ y_1(x) = u(x)$ or $y_2(x) = u(x)\, y_1(x)$. The function $u(x)$ can be found by substituting $y_2(x) = u(x)y_1(x)$ into the given differential equation.

The General Case

Suppose we divide $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ by $a_2(x)$ in order to put the equation in the **standard form**

$$y'' + P(x)y' + Q(x)y = 0$$

where $P(x)$ and $Q(x)$ are continuous on some interval *I*. Let us suppose further that $y_1(x)$ is a known solution of the standard form on *I* and that $y_1(x) \neq 0$ for every *x* in the interval. If we define $y(x) = u(x)y_1(x)$ it follows that

$$y' = uy_1' + y_1 u' \quad and \quad y'' = uy_1'' + 2y_1'u' + yu'$$

Substituting into the standard form gives

$$y'' + Py' + Qy = u\underbrace{\left(y_1'' + Py_1' + Qy_1\right)}_{=zero} + y_1u'' + \left(2y_1' + Py_1\right)u' = 0.$$

This implies that we must have

$$y_1u'' + \left(2y_1' + Py\right)u' = 0$$

Through the substitution $w = u'$ we can turn the previous equation into the following homogeneous linear D.E.

$$y_1w' + \left(2y_1' + Py\right)w = 0$$

Notice this equation is also separable.
If we separate we get

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + Pdx = 0$$

Now integrate

$$\ln|w| + 2\ln|y_1| = -\int Pdx + c$$

$$\ln\left|wy_1^2\right| = -\int Pdx + c$$

$$wy_1^2 = c_1e^{-\int Pdx}$$

$$w = u' = \frac{c_1e^{-\int Pdx}}{y_1^2}$$

If we integrate again we can find *u(x)*

$$\int \frac{c_1e^{-\int Pdx}}{y_1^2}dx$$

Lastly substituting into the original form of y_2 which was $y_2(x) = u(x)y_1(x)$ gives

$$y_2(x) = y_1(x)\int \frac{c_1e^{-\int P(x)dx}}{\left(y_1(x)\right)^2}dx$$

Ex: Given that $y_1(x) = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty,\infty)$. Use the reduction of order to find the second solution.

Ex: Given that $y_1(x) = x^3$ is a solution of $x^2y'' - 6y = 0$, use the reduction of order to find a second solution on the interval $(-\infty,\infty)$.

Ex: Given that $y_1(x) = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$, use the reduction of order to find a second solution on the interval $(0,\infty)$.

Ex: Given that $y_1\left(x\right) = \frac{\sin x}{\sqrt{x}}$ is a solution of $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$, use the reduction of order to find a second solution on the interval $(0,\infty)$.

Homogeneous Linear Equations with Constant Coefficients

For a linear 1st order D.E. $y' + ay = 0$ we can see that $y = c_1e^{ax}$ is a solution. Then we can also seek to determine whether exponential solutions exist for higher order equations that have constant coefficients.

Auxiliary Equation

Consider the special case of second order equation $ay'' + by' + cy = 0$ where a, b, and c are constants. If we try to find a solution of the form $y = e^{mx}$ then after substituting $y' = me^{mx}$ and $y'' = m^2e^{mx}$ the original equation becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0$$

since $e^{mx} \neq 0$ for all x, it is apparent the only way $y = e^{mx}$ can satisfy the D.E. is if m is chosen as a root of the quadratic equation $am^2 + bm + c = 0$. This is called the **auxiliary equation** of the differential equation.

Given that there are always two roots m_1 and m_2 to the quadratic there will be three corresponding cases

1. m_1 and m_2 are real and distinct (discriminant > 0)
2. m_1 and m_2 are real and equal (discriminant = 0)
3. m_1 and m_2 are complex conjugate numbers (discriminant < 0)

Case 1: Distinct Real Roots

Under the assumption that the auxiliary equation has two unequal real roots m_1 and m_2, we find two solutions, $y = e^{m_1 x}$ and $y = e^{m_2 x}$. We have seen that these functions are linearly independent and hence form a fundamental set. Therefore the general solution is

$$y = c_1e^{m_1 x} + c_2e^{m_2 x}$$

Case2: Repeated Real Roots

When $m_1 = m_2$, we obtain only one exponential solution, $y = e^{m_1 x}$ from the quadratic, that is $m_1 = -b/2a$ it follows from the reduction of order formula that the second solution is

$$y_2(x) = e^{m_1 x}\int \frac{e^{-(b/a)x}}{\left(e^{m_1 x}\right)^2}dx = e^{m_1 x}\int \frac{e^{2m_1 x}}{e^{2m_1 x}}dx = e^{m_1 x}\int dx = xe^{m_1 x}.$$

Therefor the general solution is

$$y = c_1e^{m_1 x} + c_2xe^{m_1 x}.$$

Case3: Conjugate Complex Roots

If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ where α and β are real and $i^2 = -1$. Formally there is no difference between this and case 1 except that we are dealing with complex numbers. So the general solution is

$$y = C_1e^{(\alpha + \beta i)x} + C_2e^{(\alpha - \beta i)x}$$

Usually in practice we prefer to work with real functions so we use Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where Θ is a real number. It follows from this formula that

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \text{ and } e^{-i\beta x} = \cos\beta x - i\sin\beta x$$

From this we can see that

$$e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x \text{ and } e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x$$

Looking back at the original general solution, if we let C_1=C_2=1 and C_1=1 and C_2=-1 we obtain the two solutions

$$y_1 = e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x} \text{ and } y_2 = e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x}.$$

Using Euler's formula these become

$$y_1 = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos\beta x \text{ and } y_2 = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin\beta x$$

Therefore the general solution $y = c_1y_1 + c_2y_2$ can be written as

$$y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right)$$

Ex: Solve the following DE's

a. $5y'' - 4y' - 12y = 0$

b. $4y'' + 4y' + y = 0$

c. $y'' + y' + y = 0$

d. $y'' + k^2y = 0$ where *k* is a real constant

Higher Order Equations

In general to solve the *nth* order D.E. where the coefficients are real constants, we must solve the *nth* degree polynomial auxiliary equation

$$a_nm^n + a_{n-1}m^{n-1} + \ldots + a_2m^2 + a_1m + a_0 = 0$$

Higher order polynomials have the three types of roots as quadratics: distinct real, repeated real, and/or complex conjugates. There are just more combinations of how these roots can come up.

If all roots are distinct reals such that $m_1 \neq m_2 \neq \ldots \neq m_n$ then the general solutions is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \ldots c_ne^{m_nx}$$

If there are repeated real roots, say m_1 has multiplicity *k,* then the general solution is

$$y = c_1e^{m_1x} + c_2xe^{m_1x} + c_3x^2e^{m_1x} + \ldots + c_kx^{k-1}e^{m_1x}$$

Complex conjugates would also occur the same as before. If a complex conjugate pair is repeated then you would for the previous repeated roots example and multiply each pair by an ascending power of *x* until you exhausted the multiplicity.

Due to the number of roots of a polynomial we can have any combinations of the previous. For example a fifth degree equation can have 3 real distinct and 2 complex, 1 distinct real a repeated real and a complex conjugate pair, etc.

Ex: Solve $y''' + 3y'' + 2y' + 6y = 0$

Ex: Solve $3y''' + 5y'' + 10y' - 4y = 0$

Ex: Solve $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$

Differential Operators

In calculus, differentiation can be denoted by the capital letter *D* that is, *dy/dx = Dy*. The symbol D is called the **differential operator** because it transforms a differentiable function into another function.

Ex: $D(\cos 4x) = -4\sin 4x$.

Higher order derivatives can be expressed in terms of D as well:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D\left(Dy\right) = D^2y$$

Where *y* represents a sufficiently differentiable function.

Polynomial expressions involving D, such as D + 3, D^2+ 3D – 4 are also differential operators. For example

$$(D+3)(5x^2+x) = (D+3)(5x^2)+(D+3)(x) = D(5x^2)+15x^2+D(x)+3x = 15x^2+13x+1$$

In general, we define an **nth order differential operator** or **polynomial operator** to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \ldots + a_1(x)D + a_0(x)$$

As a consequence of differentiation two basic properties exist for *L*:

1. *L(cf(x))= c(L(f(x)),* c is a constant
2. *L{f(x) + g(x)} = L(f(x)) + L(g(x))*

the differential operator *L* possess a linearity property; that is, *L* operating on a linear combination of two differentiable functions is the same as the linear combination L operating on the individual functions. We say *L* is a **linear operator**.

Any linear DE can be expressed in terms of D. For example $y''+5y'+6y=5x-3$ can be expressed as $D^2y+5Dy+6y=(D^2+5D+6)y=5x-3$. We can write a linear *nth* order differential equations as

L(y) = 0 or *L(y) = g(x)*

The linear differential polynomial operators can also be factored under the same rules as polynomial functions. If r_1 is a root of *L* then $(D - r_1)$ is a factor or *L*. The previous example could also be written as $(D^2+5D+6)y=(D+2)(D+3)y=5x-3$.

Annihilator Operator

If *L* is a linear differential operator with constant coefficients and *y = f (x)* is a sufficiently differentiable function such that

L(y) = 0

then *L* is said to be an **annihilator** of the function.

For example the constant function *y = k* is annihilated by *D* since *Dk = 0*. The function *y = x* is annihilated by the differential operator D^2 since *D(D(x))=D(1)=0.*

The differential operators D^n annihilates power functions up to $y = x^{n-1}$.

As an immediate consequence of this and the fact that differentiation can be done term by term, a polynomial

$$c_o + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1}$$

can be annihilated by D^n. We often want to find the differential operator of lowest order that will annihilate a function. While D^5 annihilates x^2, D^3 is the smallest one.

Ex: Find the differential operator that annihilates $2 - 6x^2 + 5x^3 - 22x^4$.

The differential operator $(D-\alpha)^n$ annihilates each of the functions

$$e^{\alpha x}, xe^{\alpha x}, x^2 e^{\alpha x}, \ldots, x^{n-1} e^{\alpha x}$$

This is due to the fact from the previous lesson where if α is a root of the auxiliary equation $(m-\alpha)^n$ is a factor and the general solution to the homogenous D.E. is

$$y = c_1 e^{\alpha x} + c_2 x e^{\alpha x} + c_3 x^2 e^{\alpha x} + \ldots + c_n x^{n-1} e^{\alpha x}$$

Ex: Find the differential operator that annihilates the given function

a. e^{-3x} **b.** $4e^{2x} - 10xe^{2x}$

The differential operator $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$ annihilates each of the functions

$$e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x, x^2 e^{\alpha x}\cos\beta x,\ \ldots, x^{n-1}e^{\alpha x}\cos\beta x$$
$$e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x, x^2 e^{\alpha x}\sin\beta x,\ \ldots, x^{n-1}e^{\alpha x}\sin\beta x$$

This can be seen by looking at the equation $[m^2 - 2\alpha m + (\alpha^2 + \beta^2)]^n = 0$ when α and π are real numbers. It has complex roots $\alpha \pm i\beta$ both of multiplicity n found by the quadratic formula.

Ex: Find the differential operator that annihilates $5e^{-x}\cos 2x - 9e^{-x} sin 2x$

When $\alpha = 0$ and $n = 1$ a special case is

$$\left(D^2 + \beta^2\right)\begin{cases}\cos\beta x \\ \sin\beta x\end{cases} = 0$$

If we want to annihilate the sum of two or more functions the differential operators L_1 and L_2 of y_1 and y_2 respectively their product $L_1 L_2$ will annihilate $c_1 y_1 + c_2 y_2$

Ex: Find the differential operator that annihilates $7 - x + 6\sin 4x$

Ex: Find the differential operator that annihilates

a. $5 + x^2 - xe^{3x}$

b. $e^{-2x} + xe^{x}$

c. $x^4 + \sin 3x - x^2 e^{5x}$

Undetermined Coefficients: Annihilator Approach

Suppose that *L(y) = g(x)* is a linear DE with constant coefficients and that *g(x)* consists of finite sums and products of the functions that we have annihilators. In other words *g(x)* is a linear combination of functions of the form

$$k \text{ (constant)},\ x^m,\ x^m e^{\alpha x},\ x^m e^{\alpha x}\cos\beta x, \text{ and } x^m e^{\alpha x}\sin\beta x$$

where *m* is a nonnegative integer and α and β are real numbers. We now know that such a function *g(x)* can be annihilated by a differential operator L_1 of lowest order consisting of a product of the operators D^n , $(D-\alpha)^n$, and $[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$.

Applying L_1 to both sides of the equation *L(y) = g(x)* yields

$$L_1L(y) = L_1g(x) = 0.$$

By solving the *homogeneous higher order* equation $L_1L(y) = 0$, we can discover the form of a particular solution y_p for the original *non-homogeneous* equation *L(y) = g(x)*.

We then substitute this assumed form into *L(y) = g(x)* to find the explicit particular solution y_p. This procedure for determining y_p, is called the **method of undetermined coefficients**.

In previous sections it was stated that the general solution of a non-homogeneous linear DE *L(y) = g(x)* is $y = y_c + y_p$, where y_c is the general solution of the associated homogeneous equation *L(y) = 0* and y_p is the particular solution of the non-homogeneous equation. Since we now know how to find both of these when the coeeficients are constants we can find the general solution to a non-homogeneous linear D.E.

Steps to Solve Undetermined Coeffieients: Annihilator Approach

If the D.E. *L(y) = g(x)* has constant coefficients, and the function *g(x)* has an differential annihilator then:

i. Find the complimentary (general) solution y_c for the associated homogeneous equation *L(y) = 0.*
ii. Apply the differential operator L_1 that annihilates the function *g(x)* on both sides of the homogeneous equation *L(y) = g(x)***.**
iii. Find the general solution of the associated higher-order homogeneous D.E. $L_1L(y) = 0$
iv. Delete from the solution in step(iii) all those terms that are duplicated in the complimentary solution y_c found in step (i). Form a linear combination y_p of the terms that remain. This is the form of a particular solution of *L(y) = g(x).*
v. Substitute y_p found in the step (iv) into *L(y) = g(x).* Match coefficients of the various functions on each side of the equality, and solve the resulting system of equations for the unknown coefficients in y_p.
vi. With the particular solution found in step (v), form the general solution $y = y_c + y_p$ of the given D.E.

Ex: Solve $y'' + 3y' + 2y = 4x^2$

Ex: Solve $y'' - 3y' = 8e^{3x} + 4\sin x$

Ex: Solve $y'' + 8y = 5x + 2e^{-x}$

Ex: Solve $y'' + y = x\cos x - \cos x$

Ex: Determine the form of a particular solution for $y'' - 2y' + y = 10e^{-2x}\cos x$

Ex: Determine the form of a particular solution for $y''' - 4y'' + 4y' = 5x^2 - 6x + 4x^2e^{2x} + 3e^{5x}$

The method of undetermined coefficients is not applicable to linear D.E. with variable coefficients nor is it applicable to linear equations with constant coefficients when g(x) is a function that does not have an annihilator such as

$$g(x) = \ln x,\ g(x) = 1/x,\ g(x) = \tan x,\ g(x) = \sin^{-1} x$$

Variation of Parameters:

It can be seen that we can find a particular solution of a linear first-order D.E. of the form $y_p = u_1(x)y_1(x)$ on an interval. Where $y_1(x)$ is a general solution to the associated homogeneous D.E. To adapt this method of **Variation of Parameters** to a linear second-order D.E. we begin by putting the equation in standard form.

$$y'' + P(x)y' + Q(x)y = f(x)$$

For the linear second-order differential equation we seek a particular solution

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where y_1 and y_2 form a fundamental set of solutions on I of the associated homogeneous D.E. therefore

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0 \text{ and } y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

We also impose the assumption $y_1u_1' + y_2u_2' = 0$ in order to simplify the first derivative and thereby the second derivate of y_p. We differentiate y_p twice giving us

$$y_p' = u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2' = u_1y_1' + u_2y_2' \text{ and } y_p'' = u_1y_1'' + y_1'u_1' + u_2y_2'' + u_2'y_2'$$

then substitute these into the original D.E. and group terms

$$u_1\left[y_1'' + Py_1' + Qy_1\right] + u_2\left[y_2'' + Py_2' + Qy_2\right] + y_1'u_1' + y_2'u_2' = f(x)$$

which gives you

$$y_1'u_1' + y_2'u_2' = f(x)$$

Now since we are going to seek two unknowns, u_1 and u_2, we need two equations. These are

$$y_1u_1' + y_2u_2' = 0 \text{ and } y_1'u_1' + y_2'u_2' = f(x)$$

From linear algebra Cramer's rule is a way of obtaining the solution of the system in terms of determinants.

Let $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, and $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$ then

$$u_1' = \frac{W_1}{W} = -\frac{y_2f(x)}{W} \text{ and } u_2' = \frac{W_2}{W} = \frac{y_1f(x)}{W}$$

Now to find u_1 and u_2 we integrate. To repeat this process for each problem will be too time consuming so it's best to just know the formulas for u_1' and u_2'. Once you have u_1 and u_2, create your particular solution $y_p = u_1y_1 + u_2y_2$ and then form your general solution

$$y = y_c + y_p$$

Ex: Solve $y'' - 4y' + 4y = (x+1)e^{2x}$
Ex: Solve $4y'' + 36y = \csc 3x$
Ex: Solve $y'' - y = \dfrac{1}{x}$

Higher-Order Equations
The same process can be used for linear nth order non-homogeneous D.E. equations put in standard form

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \dots P_1(x)y' + P_0(x)y = f(x)$$

If $y_c = c_1y_1 + c_2y_2 + \dots + c_ny_n$ is the complimentary solution to the associated homogeneous D.E., then a particular solution is $y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \dots + u_n(x)y_n(x)$
where the u_k', $k = 1, 2, \dots, 3$ are determined by the n equations

$$y_1u_1' + y_2u_2' + \dots + y_nu_n' = 0$$
$$y_1'u_1' + y_2'u_2' + \dots + y_n'u_n' = 0$$
$$\dots$$
$$y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \dots + y_n^{(n-1)}u_n' = f(x)$$

Cramer's Rule gives

$$u_k' = \frac{W_k}{W},\ k = 1, 2, \dots, n$$

where W is the Wronskian of $y_1, y_2, \dots, y_n$ and W_k is the determinant obtained by replacing the *kth* column of the Wronkskian by $(0,0,0,\dots,f(x))$

Cauchy-Euler Equation:
A linear D.E. of the form

$$a_nx^n\frac{d^ny}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1x\frac{dy}{dx} + a_0y = g(x)$$

where the coefficients $a_n, a_{n-1}, \dots, a_0$ are constants, is known as the Cauchy-Euler equation. The disguising characteristic is that the degree of x matches the order of the derivative.

We first look at the general solution to the second order homogeneous equation

$$ax^2y'' + bxy' + cy = 0$$

then the solution to higher order equations will follow.

We try a solution of the form $y = x^m$, where m is to be determined. Similar to what happened when we substituted $y = e^{mx}$, when we substitute $y = x^m$, each term of a Cauchy-Euler equation becomes a polynomial in m times x^m.

If $y = x^m$, $y' = mx^{m-1}$, and $y'' = m(m-1)x^{m-2}$ then

$$\begin{aligned} ax^2y'' + bxy' + cy &= ax^2m(m-1)x^{m-2} + bxmx^m - 1 + cx^m \\ &= am(m-1)x^m + bmx^m + cx^m \\ &= (am(m-1) + bm + c)x^m \\ &= (am^2 + (b-a)m + c)x^m = 0 \end{aligned}$$

Letting $x^m = 0$ achieves nothing, so the roots of $am^2 + (b-a)m + c = 0$, which will be the auxillary equation will give us the solutions of the D.E.

There are three different cases to consider

Case 1: Distinct Real Roots

Let m_1 and m_2 denote the real roots such that $m_1 \neq m_2$. Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ for a fundamental set of solutions. Therefore the general solution to the D.E. is

$$y = c_1x^{m_1} + c_2x^{m_2}$$

Case 2: Repeated Real Roots

If the roots are repeated, $m_1 = m_2$, we obtain one solution $y_1 = x^{m_1}$. With the discriminant being zero the root is $m_1 = \frac{-(b-a)}{2a}$. We can use the reduction of order formula to construct a second solution by first putting the second order Cauchy-Euler equation in standard form

$$y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = 0 \quad \text{where } P(x) = \frac{b}{ax} \text{ and } \int P(x)dx = \int \frac{b}{ax}dx = \ln\left(x^{\frac{b}{a}}\right)$$

So $y2 = x^{m_1}\int \frac{e^{-\ln x^{\frac{b}{a}}}}{x^{2m_1}}dx = x^{m_1}\int x^{\frac{-b}{a}}x^{-2m_1}dx = x^{m_1}\int x^{\frac{-b}{a}}x^{\frac{b-a}{a}}dx = x^{m_1}\int \frac{dx}{x} = x^{m_1}\ln x$

For higher order equations if m_1 has multiplicity k then it can be shown that

$$x^{m_1},\ x^{m_1}\ln x,\ x^{m_1}(\ln x)^2,\ x^{m_1}(\ln x)^3, \ldots\ x^{m_1}(\ln x)^{k-1}$$

are linearly independent solutions.

Case 3: Conjugate Complex Roots

If the roots are conjugate pairs $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ then the solution is

$$y = C_1x^{\alpha+i\beta} + C_2x^{\alpha-i\beta}$$

But we want to write the solution in terms of real functions only, so we use Euler's formula and get

$$y = x^{\alpha}(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x))$$

Ex: Solve $x^2y'' - 2xy' - 4y = 0$

Ex: Solve $4x^2y'' + 8xy' + y = 0$

Ex: Solve $4x^2y'' + 17y = 0$, $y(1) = -1$, $y'(1) = -\frac{1}{2}$

Ex: Solve $x^3 y''' + 5x^2 y'' + 7xy' + 8y = 0$

Ex: Solve $x^2 y'' - 3xy' + 3y = 2x^4 e^x$