Finding Limits Graphically and Numerically

Objective: Estimate a limit using a numerical or graphical approach. Learn different ways that a limit can fail to exist. Study and use a formal definition of limit.

An Intro to Limits

Sketch to graph of

$$f(x)=\frac{x^3-1}{x-1},\ x\neq 1$$

The graph is a parabola with a hole at (1,3)

Although x can not equal 1 for this function you can see what happens to f(x) as x approaches 1 from **both directions**. The notation used is

$$\lim_{x\to c} f(x)= \ or\ \lim_{x\to 1} f(x)=$$

This table shows us what is happing in the graph as well as the limit

x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
f(x)	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

Then we can say $\lim_{x\to 1} f(x)=3$

The limit must be the same from both directions!!!

Three pronged approach to problem solving (finding limits)

1. Numerical approach – Construct a table of values
2. Graphical approach – Draw a graph by hand or using technology
3. Analytic approach – Use algebra or calculus

Ex: Find the limit of f(x) as x approaches 2 where f is defined as

$$f(x)=\begin{cases}1, & x\neq 2\\ 0, & x=2\end{cases}$$

Common Types of Behavior Associated with Nonexistence of a Limit:

1. f(x) approaches a different number from the right side of c that is approaches from the left side.
2. f(x) increases or decreases without bound as x approaches c.
3. f(x) oscillates between two fixed values as x approaches c.

Some examples of limits that fail to exist

a. $\lim_{x\to 0}\frac{|x|}{x}$ **b.** $\lim_{x\to 0}\frac{1}{x^2}$ **c.** $\lim_{x\to 0}\sin\frac{1}{x}$

Evaluating Limits Analytically

Objective: Evaluate a limit using properties of limits. Develop and use a strategy for finding limits. Evaluate a limit using dividing out and rationalizing techniques. Evaluate a limit using the Squeeze Theorem.

Some Basic Limits:
Let *b* and *c* be real numbers and let *n* be a positive integer.

1. $\lim_{x \to c} b = b$ **2.** $\lim_{x \to c} x = c$ **3.** $\lim_{x \to c} x^n = c^n$

Ex: a. $\lim_{x \to 2} 3$ **b.** $\lim_{x \to -4} x$ **c.** $\lim_{x \to 2} x^2$

Properties of Limits:
Let *b* and *c* be real numbers, let *n* be a positive integer, and let *f* and *g* be functions with the following limits.

$$\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [bf(x)] = bL$

2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ provided K ≠ 0

5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Ex: a. $\lim_{x \to 3} (4x^2 - 2x + 3)(x - 1)$ b. $\lim_{x \to -2} \left(\frac{x+3}{x-2}\right)^2$

Limits of Polynomials and Rational Functions:
If p(x) and q(x) are polynomials and c is a real number, then

$$\lim_{x \to c} p(x) = p(c) \text{ and } \lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \text{ if } q(c) \neq 0$$

The Limit of a Function Containing a Radical:
Let *n* be a positive integer. The following limit is valid for all *c* if *n* is odd, and is valid for *c* > 0 if *n* is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Limits of Trigonometric Functions
Let c be a real number <u>in the domain</u> of the given trigonometric function.

1. $\lim_{x\to c} \sin x = \sin c$ **2.** $\lim_{x\to c} \cos x = \cos c$

3. $\lim_{x\to c} \tan x = \tan c$ **4.** $\lim_{x\to c} \cot x = \cot c$

5. $\lim_{x\to c} \sec x = \sec c$ **6.** $\lim_{x\to c} \csc x = \csc c$

Ex: a. $\lim_{x\to \frac{\pi}{4}} \tan x =$ **b.** $\lim_{x\to \pi} (x \sin x) =$ **c.** $\lim_{x\to 0} \cos^2 x =$

Functions That Agree at All But One Point:
Let c be a real number and let f(x) = g(x) for all x≠c in an open interval containing c. If the limit of g(x) as x approaches c exists, then the limit of f(x) also exists and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

Ex: a. $\lim_{x\to 1} \frac{x^3-1}{x-1}$ **b.** $\lim_{x\to 3} \frac{3-x}{x^2-9}$

A Strategy for Finding Limits:

1. Learn to recognize which limits can be evaluated by direct substitution

2. If the limit of f(x) as x approaches c cannot be evaluated by direct substitution, try to find a function g that agrees with f for all x other than x = c .

3. Apply the above theorem to conclude analytically that

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c)$$

4. Use a graph or table to reinforce your conclusion

Ex: a. $\lim_{x\to -3} \frac{x^2+x-6}{x-3}$ **b.** $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

Two Special Trigonometric Limits:

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ **2.** $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Ex: a. $\lim_{x\to 0} \frac{\tan x}{x}$ **b.** $\lim_{x\to 0} \frac{\sin 4x}{x}$