A Preview of Calculus
Limits and Their Properties

Objectives: Understand what calculus is and how it compares with precalculus. Understand that the tangent line problem is basic to calculus. Understand that the area problem is also basic to calculus.

Calculus is the mathematics of change - velocities and accelerations. Calculus is also the mathematics of tangent lines, slopes, areas, volumes, arc length, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

Fundamental Differences between Pre-Calc and Calc:

- An object traveling at a constant velocity can be analyzed with pre-calculus. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with pre-calculus. To analyze the slope of a curve, you need calculus.
- A tangent line to a circle can be analyzed with pre-calc. To analyze a tangent line to a general graph you need calc.
- The area of a rectangle can be analyzed with pre-calc. To analyze the area under a general curve, you need calc.

The main way we switch from pre-calc. to calc. is the use of a limit process. Calculus is a "limit machine".

Two main areas we will discuss in Calc I are:
1. The Tangent line problem
2. The Area Problem

Finding Limits Graphically and Numerically

Objective: Estimate a limit using a numerical or graphical approach. Learn different ways that a limit can fail to exist. Study and use a formal definition of limit.

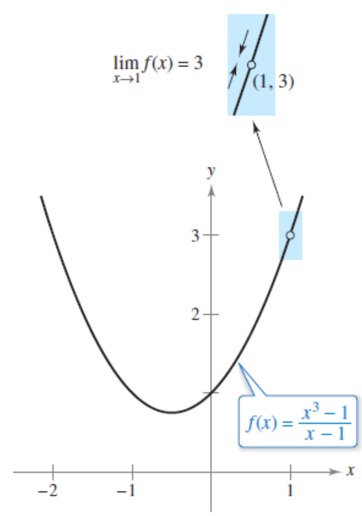
An Intro to Limits:

Sketch to graph of

$$f(x)=\frac{x^3-1}{x-1},\ x\neq 1$$

The graph is a parabola with a hole at (1,3)

Although *x* can not equal 1 for this function you can see what happens to f(x) as x approaches 1 from **both directions.**



The limit of $f(x)$ as x approaches 1 is 3.

The notation used is:

$$\lim_{x\to c} f(x) = \text{ or } \lim_{x\to 1} f(x) =$$

This table shows us what is happing in the graph as well as the limit

x	0.75	0.9	0.99	0.999	1	1.001	1.001	1.01	1.25
f(x)	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

Does it appear as though the f(x) value is approaching some finite value as x get close to 1?

Clearly from both the graph and the table the answer is yes. Then we can say

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

The limit must be the same from both directions!!!

<u>Three pronged approach to problem solving (finding limits)</u>

1. Numerical approach - Construct a table of values
2. Graphical approach - Draw a graph by hand or using technology
3. Analytic approach - Use algebra or calculus

Common Types of Behavior Associated with Nonexistence of a Limit:

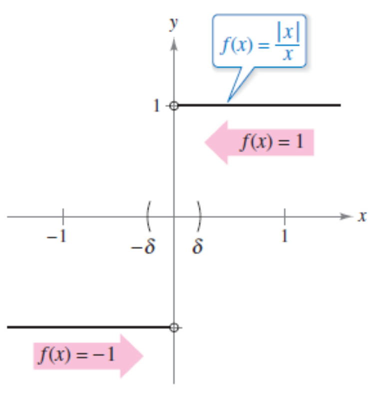
1. f(x) approaches a different number from the right side of c that is approaches from the left side.
2. f(x) increases or decreases without bound as x approaches c.
3. f(x) oscillates between two fixed values as x approaches c.

Some examples of limits that fail to exist

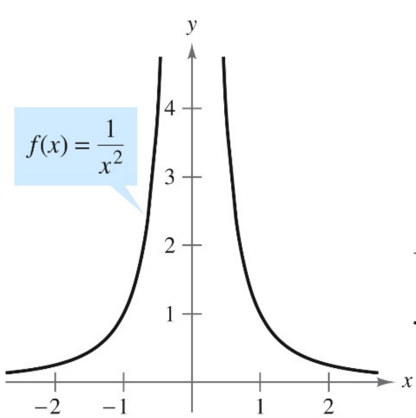
a. $\lim_{x \to 0} \frac{|x|}{x}$

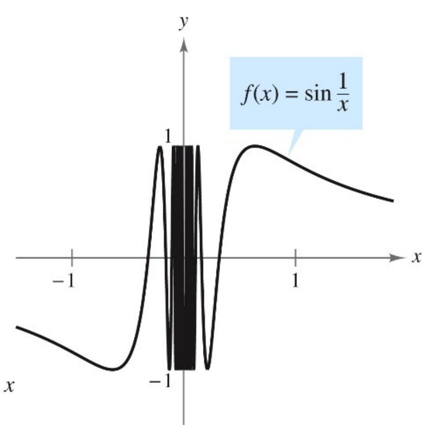
b. $\lim_{x \to 0} \frac{1}{x^2}$

c. $\lim_{x \to 0} \sin \frac{1}{x}$



$\lim_{x \to 0} f(x)$ does not exist.





A Formal Definition of Limit:
Let f be a function defined on an interval containing c (except possibly at c) and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each ε > 0 there exists a δ > 0 such that if $|x - c| < \delta$ then $|f(x) - L| < \varepsilon$

Ex: Use the formal definition of a limit to prove $\lim_{x \to 2} 3x - 5 = 1$
Solution: You must show that for each ε > 0 there exists a δ > 0 such that if

$$|x - 2| < \delta \text{ then } |(3x - 5) - 1| < \varepsilon$$

Here we need to work with the $|f(x) - L|$ and relate it to $|x - c|$ to see the relationship between ε and δ.

This is how the proof should be written formally:
Given ε let δ=ε/3 then

$$\begin{aligned} |x - c| < \delta &\Rightarrow |x - 2| < \varepsilon/3 \\ &\Rightarrow 3|x - 2| < \varepsilon \\ &\Rightarrow |3x - 6| < \varepsilon \\ &\Rightarrow |3x - 5 - 1| < \varepsilon \\ &\Rightarrow |f(x) - L| < \varepsilon \end{aligned}$$

Q.E.D.

Evaluating Limits Analytically

Objective: Evaluate a limit using properties of limits. Develop and use a strategy for finding limits. Evaluate a limit using dividing out and rationalizing techniques. Evaluate a limit using the Squeeze Theorem.

The limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$.
In such cases, the limit can be evaluated by **direct substitution**. That is,

$$\lim_{x \to c} f(x) = f(c)$$

Such *well-behaved* functions are **continuous at *c***.

Through the formal definition of limits we can easily see some simple limits can be evaluated through this direct substitution method

Some Basic Limits:

Let b and c be real numbers

$$\lim_{x \to c} b = b \qquad \lim_{x \to c} x = c$$

Ex: Evaluate the limits

a. $\lim_{x \to 2} 3 =$ b. $\lim_{x \to 3} x =$

Properties of Limits:

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits

$$\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} g(x) = K$$

then the following properties for limits can apply (can be proven using the formal def'n).

- Scalar multiple: $\lim_{x \to c} bf(x) = bL$
- Sum and Differences: $\lim_{x \to c} \left[f(x) \pm g(x) \right] = L \pm K$
- Product: $\lim_{x \to c} f(x)g(x) = LK$
- Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}, \quad \text{as long as } \lim_{x \to c} g(x) = K \neq 0$
- Power: $\lim_{x \to c} \left(f(x) \right)^n = L^n$

Ex: Evaluate the limit using the proceeding properties, if possible.

a. $\lim_{x\to 2} 4x^2+3$

b. $\lim_{x\to 1} \frac{x^2+x+2}{x+1}$

Limits of Polynomials and Rational Functions:

If *p(x)* and *q(x)* are polynomials and *c* is a real number, then

$$\lim_{x\to c} p(x) = p(c) \text{ and } \lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}, \text{ as long as } q(c)\neq 0$$

The Limit of a Function Containing a Radical:

Let *n* be a positive integer. The following limit is valid for all *c* if *n* is odd, and is valid for *c* > 0 if *n* is even.

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Limits of Trigonometric Functions:

Let *c* be a real number in the domain of the given trigonometric function.

$\lim_{x\to c} \sin x = \sin c$ $\qquad$ $\lim_{x\to c} \csc x = \csc c$

$\lim_{x\to c} \cos x = \cos c$ $\qquad$ $\lim_{x\to c} \sec x = \sec c$

$\lim_{x\to c} \tan x = \tan c$ $\qquad$ $\lim_{x\to c} \cot x = \cot c$

Ex: Evaluate the limit analytically, if it exists

a. $\lim_{x\to 0} \tan x$ $\qquad$ b. $\lim_{x\to \pi} x\cos x$ $\qquad$ c. $\lim_{x\to 0} \sin^2 x$

It's not always this easy! Most limits you come across will not be done with direct substitution.

Functions That Agree at All But One Point:

> Let *c* be a real number and let *f(x) = g(x)* for all $x \neq c$ in an open interval containing *c*. If the limit of *g(x)* as *x* approaches *c* exists, then the limit of *f(x)* also exists and
> $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

Ex: Find the limit analytically, it is exists

a. $\lim_{x\to 1}\frac{x^3-1}{x-1}$

b. $\lim_{x\to 2}\frac{2-x}{x^2-4}$

A Strategy for Finding Limits:

> 1. Learn to recognize which limits can be evaluated by direct substitution
> 2. If the limit of *f(x)* as *x* approaches *c* cannot be evaluated by direct substitution, try to find a function *g* that agrees with *f* for all *x* other than *x = c*.
> 3. Apply the previous theorem to conclude analytically that
> $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c)$$
> 4. Use a graph or table to reinforce your conclusion

Ex: Find the limit analytically, if it exists

a. $\lim_{x\to -3}\frac{x^2+x-6}{x+3}$

b. $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

These simple technics from algebra (factoring, canceling, rationalizing) don't always work either!

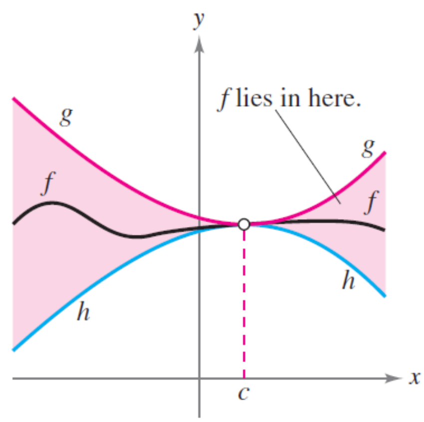
The Squeeze Theorem:

For $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = \lim_{x \to c} g(x) = L$$

then $\lim_{x \to c} f(x)$ exists and is also equal to L.

$h(x) \le f(x) \le g(x)$



The Squeeze Theorem

Two Special Trigonometric Limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Ex: Evaluate the limit analytically, if it exists

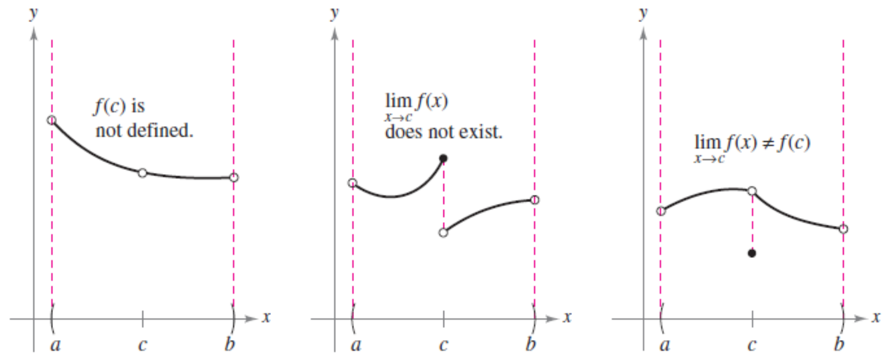
a. $\lim_{x \to 0} \dfrac{\sin x(1 - \cos x)}{x^2}$

b. $\lim_{x \to 0} \dfrac{\tan 5x}{x}$

Continuity and One-Sided Limits:

Objective: Determine continuity at a point and continuity on an open interval. Determine one-sided limits and continuity on a closed interval. Use properties of continuity.

In mathematics, the term *continuous* has much the same meaning as it has in everyday usage. Informally, to say that a function f is continuous at $x = c$ means that there is no interruption in the graph of f at c. That is, its graph is unbroken at c and there are no holes, jumps, or gaps.



Three conditions exist for which the graph of f is not continuous at $x = c$.

Continuity at a Point:

A function f is continuous at c if the following three conditions are met.

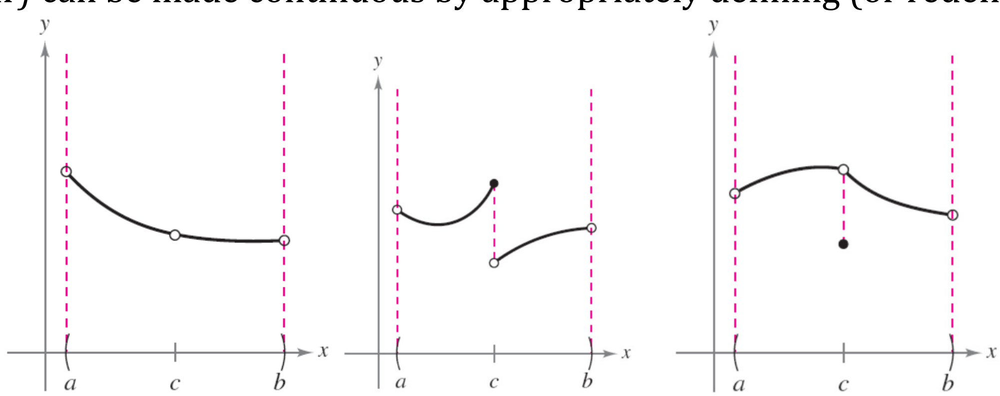
1. f(c) is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

Ex: Show that $f(x) = 3x + 2$ is continuous at x = 2

Continuity on an Open Interval:

A function is continuous on an open interval (a,b) if it is continuous at each point in the interval. A function that is continuous on the entire real line is everywhere continuous.

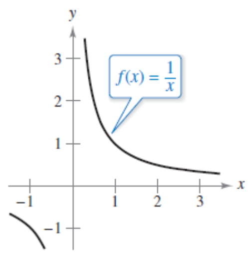
If f is not continuous at $x = c$ then f is said to have a **discontinuity** at c. Discontinuities fall into 2 categories: **Removable** and **Unremovable**. A discontinuity at c is called removable if f can be made continuous by appropriately defining (or redefining $f(c)$).



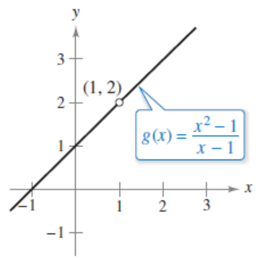
(a) Removable discontinuity **(b)** Nonremovable discontinuity **(c)** Removable discontinuity

Ex: Look at the following:

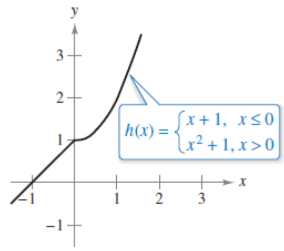
a. $f(x)=\dfrac{1}{x}$ b. $g(x)=\dfrac{x^2-1}{x-1}$ c. $h(x)=\begin{cases} x+1, & x\le 0 \\ x^2+1, & x>0\end{cases}$ d. $y=\sin x$



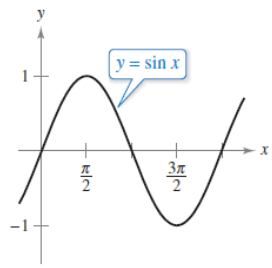
(a) Nonremovable discontinuity at $x = 0$



(b) Removable discontinuity at $x = 1$



(c) Continuous on entire real number line

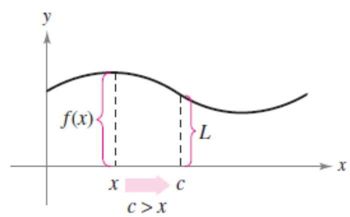


(d) Continuous on entire real number line

One Sided Limits:

If *f(x)* approaches *L* as *x* tends toward *c* from the left ($x < c$), we write

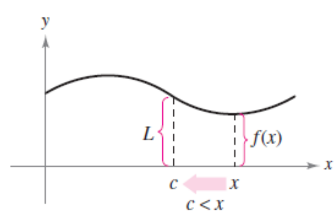
$$\lim_{x\to c^-} f(x) = L$$



(b) Limit as x approaches c from the left.

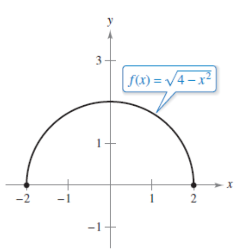
if *f(x)* approaches *M* as *x* tends toward *c* from the right ($c < x$), then we write

$$\lim_{x\to c^+} f(x) = M$$



(a) Limit as x approaches c from the right.

Ex: Evaluate $\lim\limits_{x\to -2^+} \sqrt{4-x^2}$ if it exists.



The limit of $f(x)$ as x approaches -2 from the right is 0.

Existence of a Limit (Alternative Definition):

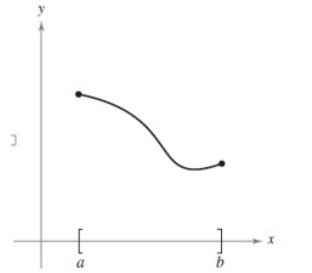
Let *f* be a real function and let *L* and *c* be real numbers. The limit of *f(x)* as *x* approaches *c* is *L* if and only if (iff)

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

Continuity on a Closed Interval:

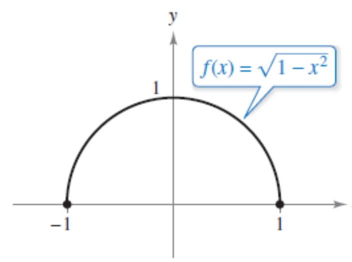
A function *f* is continuous on the closed interval [*a*,*b*] if it is continuous on the open interval (*a*,*b*) and

$$\lim_{x\to a^+} f(x) = f(a) \text{ and } \lim_{x\to b^-} f(x) = f(b)$$



Continuous function on a closed interval

Ex: Look at closed interval continuity of $f(x)=\sqrt{1-x^2}$



f is continuous on [−1, 1].

Infinite Limits

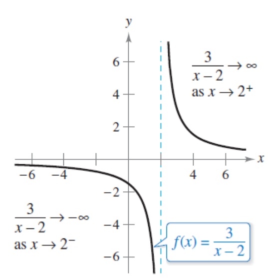
Objective: Determine infinite limits from the left and from the right. Find and sketch the vertical asymptotes of the graph of a function.

Vertical Asymptotes (Definition from Pre-Calc)

If *f(x)* approaches infinity (or negative infinity) as x approaches *c* from the right or left, then *x* = *c* is a vertical asymptote of the graph of *f*.

The best way to find a vertical asymptote for a simple rational function is to find all the values for which the denominators are equal to zero but the numerators are NOT.

Ex: Look at the function $f(x)=\dfrac{3}{x-2}$ if x = 2 the denominator is zero but not the numerator



f(*x*) increases and decreases without bound as *x* approaches 2.

Ex: Find the vertical asymptotes for the following

a. $f(x)=\dfrac{2x}{x+1}$ b. $g(x)=\dfrac{x-1}{x^2-1}$ c. $h(x)=\dfrac{1}{x^2+9}$

Infinite Limits:

Let f be a function that is defined at every real number in some interval containing c (except possibly at c itself). The statement

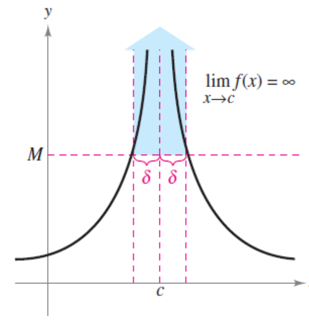
$$\lim_{x \to c} f(x) = \infty$$

means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$.

Similiarly, the statement

$$\lim_{x \to c} f(x) = -\infty$$

means that for each $N > 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$.



In other words a limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. These occur at vertical asymptotes.

Ex: Evaluate the limits, if they exists

a. $\displaystyle \lim_{x \to 1} \frac{1}{(x-1)^2}$ b. $\displaystyle \lim_{x \to 1^-} \frac{-1}{x-1}$ c. $\displaystyle \lim_{x \to 1} \frac{-1}{x-1}$