Integration Techniques
Integration by Parts

This technique is particularly useful for integrands involving products of algebraic functions and transcendental functions. The formula is based on the product rule for derivatives.

Integration by Parts
If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du$$

Guidelines for Integration by Parts:
- Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.
- Try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.

Ex: **a.** $\int xe^{x}dx$ **b.** $\int 3x^{4}\ln dx$

c. $\int \ln x\,dx$ **c.** $\int_{0}^{1} \arcsin x\,dx$

Repeated use of integration by parts: Sometimes you need to apply IBP multiple times

Ex: $\int x^{3}\cos x\,dx$

Tabular Method

This can be used to help organize your work. This method works well for integrands of the form $x^{n}\sin ax, x^{n}cosax, x^{n}e^{ax}$

Ex: $\int x^{2}\sin 4x\,dx$

Ex: $\int \sec^3 x dx$

Summary of Common Integrals using Integration by Parts

1. For integrals of the form

$$\int x^n e^{ax} dx, \int x^n \sin ax dx, \int x^n \cos ax dx$$

let $u = x^n$ and let $dv = e^{ax} dx$, $\sin ax dx$, or $\cos ax dx$

2. for integrals of the form

$$\int x^n \ln x dx, \int x^n \arcsin ax dx, \int x^n \arctan ax dx$$

let $u =$ *lnx*, *arcsinax*, or *arctanax* and let $dv = x^n dx$

3. For integrals of the form

$$\int e^{ax} \sin bx dx \text{ or } \int e^{ax} \cos bx dx$$

let $u = \sin bx$ or $\cos bx$ and let $dv = e^{ax} dx$

Ex: $\int e^x \sin x dx$

Trigonometric Integrals

This section we study techniques for integrals of the form

$$\int \sin^m x \cos^n x dx \text{ and } \int \sec^m x \tan^n x dx$$

The technique we use stems from examples seen back in Calc 1 using u-substitution**.**

Ex: $\int \sin^5 x \cos x dx$

To break up identities $\int \sin^m x \cos^n x dx$ into forms that use power rule try using the following identities

$$\sin^2 x + \cos^2 x = 1, \quad \sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

<u>Guidelines for Evaluating Integrals Involving Sine and Cosine</u>

1. If the power of the sin if odd and positive, save one sin factor and convert the remaining factors to cos. Then, expand and integrate.

$$\int \sin^{2k+1} x \cos^m x dx = \int \cos^m x (\sin^2 x)^k \sin x dx = \int \cos^m x (1-\cos^2 x)^k \sin x dx$$

2. If the power of the cos is odd and positive, save one cos factor and convert the remaining factors to sin. Then, expand and integrate.

$$\int \sin^m x \cos^{2k+1} x dx = \int \sin^m x (\cos^2 x)^k \cos x dx = \int \sin^m x (1-\sin^2 x)^k \cos x dx$$

3. If the powers of both the sin and cos are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1-\cos 2x}{2} \text{ and } \cos^2 x = \frac{1+\cos 2x}{2}$$

to convert the integrand to odd powers of the cos. Then proceed to #2.

Ex: **a.** $\int \sin^3 x \cos^4 x dx$ **b.** $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx$

Ex: a. $\int \cos^4 x dx$ **b.** $\int_0^{\pi/2} \cos^4 x dx$

Wallis's Formula

1. If n is odd $(n \geq 3)$, then

$$\int_0^{\pi/2} \cos^n x dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{n-1}{n}\right)$$

2. If n is even $(n \geq 2)$, then

$$\int_0^{\pi/2} \cos^n x dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

Integrals Involving Powers of Secant and Tangent: $\int \sec^m x \tan^n x dx$

<u>Guidelines for Evaluating Integrals Involving Secant and Tangent</u>

1. If the power of sec is even and positive, save a sec-squared factor and convert the remaining factors to tan. Then expand and integrate.

$$\int \sec^{2k} \tan^n x dx = \int (\sec^2 x)^{k-1} \tan^n x \sec^2 x dx = \int (1+\tan^2 x)^{k-1} \tan^n x \sec^2 x dx$$

2. If the power of the tan is odd and positive, save a sec-tan factor and convert the remaining factors to sec. Then expand and integrate.

$$\int \sec^m x \tan^{2k+1} x dx = \int \sec^{m-1} x (\tan^2 x)^k \sec x \tan x dx = \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x dx$$

3. If there are no sec factors and the power of the tan is even and positive, convert a tan2 factor to a sec2 factor, then expand and repeat if necessary.

$$\int \tan^n x dx = \int \tan^{n-2} x (\tan^2 x) dx = \int \tan^{n-2} x (\sec^2 x - 1) dx$$

4. If the integral is of the form $\int \sec^m x dx$ where m is odd and positive, use integration by parts.

5. If none of the first four guidelines applies, try converting to sin and cos.

Ex: a. $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$

b. $\int \sec^4 3x \tan^3 x dx$

c. $\int_0^{\pi/4} \tan^4 x dx$

d. $\int \frac{\sec x}{\tan^2 x} dx$

<u>Integrals Involving Sin-Cos Products with Different Angles</u>

Product -Sum Identities:

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$$

Ex: $\int \sin 5x \cos 4x dx$

Trigonometric Substitution

We will use trig-substitution to evaluate integrals involving

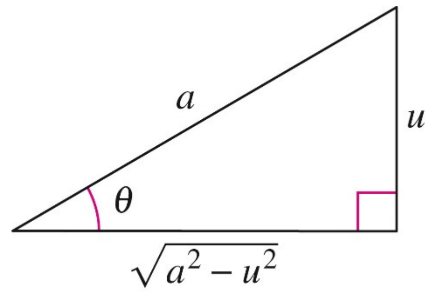
$$\sqrt{a^2-u^2}\ ,\ \sqrt{a^2+u^2}\ ,\ \sqrt{u^2-a^2}$$

We will also use the Pythagorean Identities

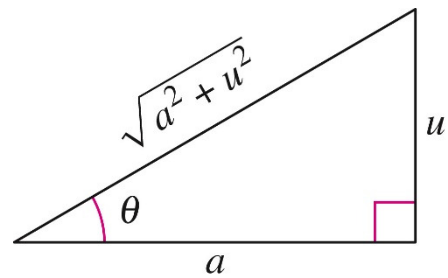
$$\cos^2 x = 1-\sin^2 x,\quad \sec^2 x = 1+\tan^2 x,\quad \tan^2 x = \sec^2 x - 1$$

Trigonometric Substitution (a > 0)

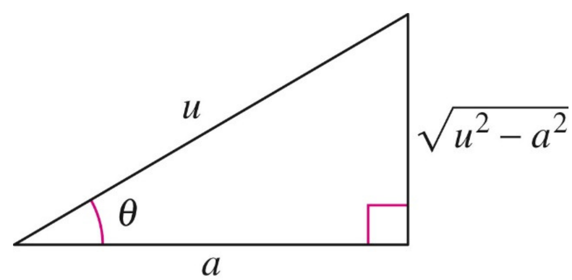
For integrals involving $\sqrt{a^2-u^2}$,
let $u = a\sin\theta$, then $\sqrt{a^2-u^2} = a\cos\theta$,
where $-\pi/2 \le \theta \le \pi/2$



For integrals involving $\sqrt{a^2+u^2}$,
let $u = a\tan\theta$, then $\sqrt{a^2+u^2} = a\sec\theta$,
where $-\pi/2 \le \theta \le \pi/2$



For integrals involving $\sqrt{u^2-a^2}$,
let $u = a\sec\theta$, then $\sqrt{u^2-a^2} = \pm a\tan\theta$
where $0 \le \theta \le \pi/2, \pi/2 \le \theta \le \pi$
(use + if u > a and − if u < -a.
This can only be done for definite integrals)



Ex: a. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

b. $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$

c. $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$

d. $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$

Special Integration Formula's (a > 0)

- $\int \sqrt{a^2 - u^2}\,du = \frac{1}{2}\left(a^2 \arcsin\frac{u}{a} + u\sqrt{a^2 - u^2} \right) + C$
- $\int \sqrt{u^2 - a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln|u + \sqrt{u^2 - a^2}| \right) + C, \qquad u > a$
- $\int \sqrt{u^2 + a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2 + a^2} - a^2 \ln|u + \sqrt{u^2 + a^2}| \right) + C$

Ex: $\int \sqrt{4 - x^2}\,dx$

Partial Fractions

We will use Partial Fractions to break rational functions into simpler rational functions in order to integrate.

Consider $\int \frac{1}{x^2 - 5x + 6}dx$ we could use Trig Sub. but if the denominator can be factored we can use the algebra technique partial fraction decomposition.

Decomposition of N(x)/D(x) into Partial Fractions:

1. **Divide if improper**: If N(x)/D(x) is an improper fraction divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (a\ polynomial) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of D(x).

2. **Factor denominator**: Completely factor the denominator into factors of the form

$$\left(px + q\right)^m \ \ and\ \ \left(ax^2 + bx + c\right)^n$$

3. **Linear factors**: For each factor of the $\left(px + q\right)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px+q)} + \frac{A_2}{(px+q)^2} + \ldots + \frac{A_m}{(px+q)^m}$$

4. **Quadratic Factors**: For each factor of the form $\left(ax^2 + bx + c\right)^n$, partial fraction decomposition must include the following sum of n factors.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{A_2}{(ax^2 + bx + c)^2} + \ldots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Ex: Write the partial fraction decomposition for $\frac{1}{x^2 - 5x + 6}$

Ex: a. $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

b. $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx$

c. $\int \frac{8x^3+13x}{(x+2)^2}\,dx$

Guidelines for solving the Basic Equation

<u>Linear Factors</u>

1. Substitute the roots of the distinct linear factors into the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then sub other convenient values of x and solve for remaining coefficients.

<u>Quadratic Factors</u>

1. Expand the basic equation
2. Collect terms according to powers of x.
3. Equate the coefficients of like powers to obtain a system of linear equations involving A, B, C, and so on.
4. Solve the system of linear equations.

Indeterminate Forms and L'Hopital's Rule

0/0 and ∞/∞ are called indeterminate forms. It may appear that a limit is indeterminate but that may not guarantee the limit doesn't exist.

Take a look at the following two limits

$$\lim_{x\to 0}\frac{e^{2x}-1}{e^{x}-1} \text{ and } \lim_{x\to 0}\frac{e^{2x}-1}{x}$$

To find some of these limits you can use L'Hopital's Rule

L'Hopital's Rule:

Let f and g be functions that are differentiable on an open interval (a,b) containing c except possibly at c itself. Assume that $g'(x)\neq 0$ for all x in (a,b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form of 0/0, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit of the right exists (or is infinite). This result also applies if the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms

$$\infty/\infty, -\infty/\infty, \infty/-\infty, \text{ or } -\infty/-\infty.$$

Other indeterminate forms are: $\infty-\infty$, $0\bullet\infty$, 0^0, 1^∞, ∞^0

Ex: **a.** $\displaystyle\lim_{x\to 0}\frac{e^{2x}-1}{x}$ **b.** $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}$ **c.** $\displaystyle\lim_{x\to\infty}\frac{x^2}{e^{-x}}$

d. $\displaystyle\lim_{x\to\infty}e^{-x}\sqrt{x}$ **e.** $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ **f.** $\displaystyle\lim_{x\to 0^+}(\sin x)^x$

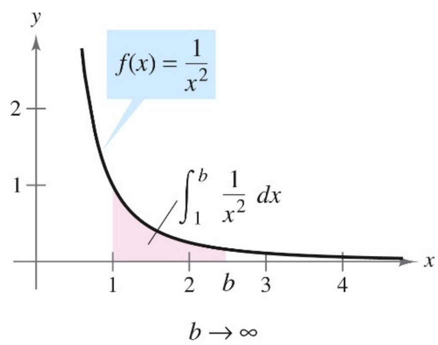
g. $\displaystyle\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

The following are determinant: $\infty+\infty\to\infty$, $-\infty-\infty\to-\infty$, $0^\infty\to 0$, $0^{-\infty}\to\infty$

Improper Integrals

Definition of a definite integral $\int_a^b f(x)dx = F(b) - F(a)$ requires

- $[a,b]$ to be finite, and
- f must be continuous on $[a,b]$.



This section we will look at integrals that don't satisfy these requirements. Either the limits of integration are finite or the function has a number of infinite discontinuities. Integrals that possess either of these properties are called **improper integrals**.

To get the idea look at $\int_1^b \frac{dx}{x^2}$ as b approaches infinity.

Definition of Improper Integrals with Infinite Integration Limits

- If f is continuous on the interval $[a,\infty)$, then

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

- If f is continuous on the interval $(-\infty,b]$, then

$$\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$$

- If f is continuous on the interval $(-\infty,\infty)$, then

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

where c is any real number.

In the first two cases, the improper integral **converges** if the limit exists otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

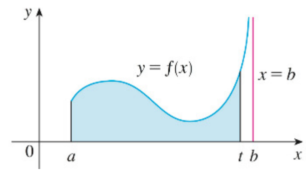
Ex: **a.** $\int_1^\infty \frac{dx}{x}$ **b.** $\int_0^\infty e^{-x}dx$ **c.** $\int_0^\infty \frac{1}{x^2+1}dx$

d. $\int_1^\infty (1-x)e^{-x}dx$ **e.** $\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}dx$

Definitions of Improper Integrals with Infinite Discontinuities

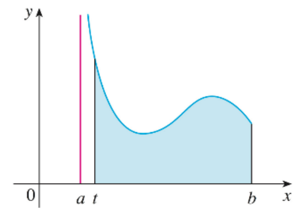
- If f is continuous on the interval $[a,b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$



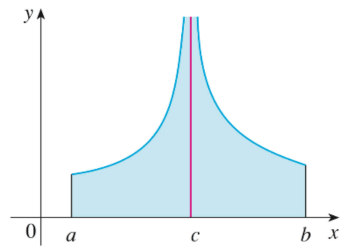
- If f is continuous on the interval $(a,b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)dx = \lim_{c\to a^+}\int_c^b f(x)dx$$



- If f is continuous on the interval [a,b], except at some c in (a,b) at which has an infinite discontinuity, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



In the first two cases, the improper integral **converges** if the limit exists otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverge.

Ex: a. $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$ **b.** $\displaystyle\int_0^2 \frac{dx}{x^3}$ **c.** $\displaystyle\int_{-1}^2 \frac{dx}{x^3}$

d. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

A Special Type of Improper Integral

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & if\ p>1 \\ diverges, & if\ p\le 1 \end{cases}$$

Ex: $\displaystyle\int_1^\infty \frac{dx}{x}$