Infinite Series

Sequences:

A sequence in defined as a function whose domain is the set of positive integers. Usually it's easier to denote a sequence in subscript form rather than function notation.

- $a_1, a_2, a_3, \ldots a_n$ are the **terms** of the sequence and
- a_n is the **nth term**

Listing Terms of a Sequence

a. $\{ a_n \} = \{3 + (-1)^n\}$ **b.** $\{ b_n \} = \{\frac{n}{1-2n}\}$

c. a **recursively defined** sequence $\{ c_n \}$, where $c_1 = 25$ and $c_{n+1} = c_n - 5$.

Pattern Recognition for Sequences

Ex: Find the sequence $\{a_n\}$ first five terms are $\left\{\frac{2}{1},\frac{4}{3},\frac{8}{5},\frac{16}{7},\frac{32}{9},\ldots\right\}$

Ex: Determine the nth term of the sequence whose first five terms are

$$\left\{-\frac{2}{1},\frac{8}{2},-\frac{26}{6},\frac{80}{24},-\frac{242}{120},\ldots\right\}$$

Limit of a Sequence

If the terms of a sequence approach a limiting value the sequence is said to **converge.**

Def'n of the Limit of a Sequence

Let L be a real number. The limit of a sequence $\{ a_n \}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

if for each ε > 0, there exists M > 0 such that $|a_n - L| < \varepsilon$ whenever n > M. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges**.

Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{n\to\infty} f(x) = L$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L$$

Ex: Find the limit of the sequence whose nth term is $a_n = \left(1+\frac{1}{n}\right)^n$

Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$

1. $\lim_{n\to\infty}(a_n \pm b_n) = L \pm K$ **2.** $\lim_{n\to\infty} ca_n = cL$, c is any real number

3. $\lim_{n\to\infty} a_n b_n = LK$ **4.** $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

Ex: a. Find the limit of $\{a_n\} = \{3 + (-1)^n\}$ if it exists.

b. Find the limit of $\{b_n\} = \{\frac{n}{1-2n}\}$ if it exists.

Ex: Show that the sequence whose nth term is $a_n = \frac{n^2}{2^n - 1}$ converges

Squeeze Theorem for Sequences

If $\lim_{x\to\infty} a_n = L = \lim_{x\to\infty} b_n$

and there exists an integer N such that $a_n < c_n < b_n$ for all $n > N$, then $\lim_{x\to\infty} c_n = L$

Absolute Value Theorem

For a sequence $\{a_n\}$, if

$\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

Ex: Show that $\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$ converges, and find its limit.

Monotonic Sequences and Bounded Sequences

Definition of a Monotonic Sequence

A sequence $\{a_n\}$ is monotonic if all its terms are non-decreasing or non-increasing

$$a_1 \le a_2 \le a_3 \le \ldots \le a_n \text{ or } a_1 \ge a_2 \ge a_3 \ge \ldots \ge a_n$$

Ex: Determine whether each sequence is monotonic

a. $a_n = 3 + (-1)^n$ **b.** $b_n = \frac{2n}{1+n}$ **c.** $c_n = \frac{n^2}{2^n - 1}$

Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \le M$ for all n. The number M is called an **upper bound** of the sequence.

2. A sequence $\{a_n\}$ is **bounded below** if there is a real number M such that $N \le a_n$ for all n. The number N is called an **lower bound** of the sequence.

3. A sequence $\{a_n\}$ is bounded if it is bounded above and below.

Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Ex: a. The sequence $\{a_n\} = \{1/n\}$ is both bounded and monotonic and so, by the last Thm must converge.

b. The divergent sequence $\{b_n\} = \{n^2/(n+1)\}$ is monotonic, but not bounded. (its only bounded below)

c. the divergent sequence $\{c_n\} = \{(-1)^n\}$ is bounded but not monotonic.

Series and Convergence

If $\{a_n\}$ is an infinite sequence then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ...$$ is an **infinite series**.

Def'n of Convergent and Divergent Series

For infinite series $\sum_{n=1}^{\infty} a_n$, the **nth partial sum** is given by

$S_n = a_1 + a_2 + \ldots + a_n$.

- If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges**. The limit S is called the sum of the series.

  $$S = a_1 + a_2 + \ldots + a_n + \ldots$$

- If $\{S_n\}$ diverges, then the series **diverges**.

Ex: a. $\sum_{n=1}^{\infty} \frac{1}{2^n}$ **b.** $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ **c.** $\sum_{n=1}^{\infty} 1$

The series in example b is a **telescoping series** of the form $(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \ldots$ the nth partial sum is $S_n = b_1 - b_{n+1}$ is follows that a telescoping series converges iff b_{n+1} approaches a finite number as $n \to \infty$.

If the series converges then $S = b_1 - \lim_{n\to\infty} b_{n+1}$.

Ex: $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$

Geometric Series

$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots + ar^n + \ldots, \ a \neq 0$ is a **geometric series** with ratio r.

Convergence of a Geometric Series

A geometric series with ratio r diverges if $|r| \geq 1$.

If $0 < |r| < 1$, then the the series converges and $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Ex: a. $\sum_{n=0}^{\infty} \frac{3}{2^n}$ **b.** $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

Properties of Infinite Series

If $\sum a_n = A$, $\sum b_n = B$ and c is any real number, then the following series converges to the indicated sums

1. $\sum_{n=1}^{\infty} ca_n = cA$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

nth – Term Test for Divergence

The following thm states that if a series converges, the limit of its nth term must be 0.

Limit of nth Term of Convergent Series

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

nth – Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex: a. $\sum_{n=0}^{\infty} 2^n$ **b.** $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$ **c.** $\sum_{n=1}^{\infty} \frac{1}{n}$

The Integral Test and *p*-Series

The Integral Test

If f is a positive, continuous, and decreasing for x ≥ 1 and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx$$

either both converge or both diverge.

Ex: a. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ **b.** $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

p – Series and Harmonic Series

$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is called a **p–series** where p is a positive constant.

If p = 1, $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is called a **Harmonic series.**

A general harmonic series is of the form Σ1/(an+b).

Convergence of a p - Series

The p – series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ...$$

1. converges if p > 1, and
2. diverges if 0 < p < 1.

Ex: Determine whether the following series converges or diverges:

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Comparisons of Series

Direct Comparison Test

Let $0 < a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Ex: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$.

Ex: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$.

Limit Comparison Test

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n} \right) = L$$

where L is finite and positive. Then the two series Σa_n and Σb_n either both converge of both diverge.

Ex: Show that the following general harmonic series diverges.

$$\sum_{n=1}^{\infty} \frac{1}{an+b}, \ a>0, \ b>0$$

Ex: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$.

Ex: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$.

Alternating Series

A series where the terms continuously switch from positive to negative or vice versa is known as an **alternating series**.

Ex: $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2^n}$

Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \text{ and } \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \le a_n, \text{ for all } n$

Ex: Determine the convergence or divergence of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

Ex: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$

Ex: Does the Alternating Series Test apply to $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$

Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \le a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \le a_{N+1}$$

Ex: Approximate the sum of the following series by its first six terms.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n!}\right)$$

Absolute Convergence

If the series $\sum |a_n|$ converges, then $\sum a_n$ also converges.

Def'n of Absolute Value and Conditional Convergence

1. $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.
2. $\sum a_n$ is **conditionally convergent** if $\sum a_n$ converges and $\sum |a_n|$ diverges.

Ex: Determine whether each of the series is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n n!}{2^n}$ b. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ c. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n(n+1)/2}}{3^n}$ d. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n+1)}$

The Ratio and Root Tests

Ratio Test

Let Σa_n be a series with nonzero terms.

1. Σa_n converges absolutely if $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$

2. Σa_n diverges if $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$

3. The Ratio Test is inconclusive if $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$

Ex: Determine the convergence or divergence of $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n!}$

Ex: Determine whether each series converges or diverges.

a. $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **b.** $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ **c.** $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

Root Test

Let Σa_n be a series

1. Σa_n converges absolutely if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

2. Σa_n diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

3. The Root Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Ex: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$.

Guidelines for Testing a Series for Convergence or Divergence.

1. Does the nth term approach 0? If not the series is divergent.

2. Is the series one of the special types: geometric, p-series, telescoping, or alternating?

3. Can the Integral Test, Root Test, or Ratio Test be applied?

4. Can the series be compared favorably to one of the special types?

Determine the convergence or divergence of each series.

a. $\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$ **b.** $\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$ **c.** $\sum_{n=1}^{\infty} ne^{-n^2}$

d. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ **e.** $\sum_{n=1}^{\infty} (-1)^n \frac{3}{4n+1}$ **f.** $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ **g.** $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+1}\right)^n$

Power Series

Def'n of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n + \ldots$$

is called a **power series**. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots a_n(x-c)^n + \ldots$$

is called a **power series centered at *c***, where *c* is a constant.

Ex: Find the center of the following power series

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ **b.** $\sum_{n=0}^{\infty} (-1)(x+1)^n$ **c.** $\sum_{n=0}^{\infty} \frac{1}{n}(x-1)^n$

Radius and Interval of Convergence.

A power series can be thought of as a function f where the domain of f is the set of all values x for which the power series converges.

The series always converges at its center c, so c always lies in the domain of f.

Convergence of Power Series

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.

2. There exists a real number R > 0 such that the series converges absolutely for $|x-c| < R$, and diverges for $|x-c| > R$.

3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series.

- If the series converges only at c, the radius of convergence is R = 0
- If the series converges for all x, the radius of convergence if R = ∞.

The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

Ex: Find the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$

Ex: Find the radius of convergence of $\sum_{n=0}^{\infty} 3(x-2)^n$

Ex: Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Endpoint Convergence

When r is a finite number the previous thm says nothing about the convergence at the endpoints of the interval (there is no or equal to) so they need to be tested separately.

Ex: Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n}$

Ex: Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$

Ex: Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

Differentiation and Integration of Power Series.

Power series representation of functions has played an important role in Calculus. Much on Newtons work with differentiation an integration was done in the context of power series.

Properties of Functions Defined by Power Series

If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$$

has a radius of convergence R > 0, then, on the interval (c – R, c + R), f is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

1. $f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$

$$= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \ldots$$

2. $\int f(x)dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$

$$= C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + \ldots$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series if the same as that of the original power series. The *interval of convergence*, however, may be different as a result of the behavior at the endpoints.

Ex: For $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

Find the intervals of convergence for each of the following

a. $\int f(x)dx$ **b.** $f(x)$ **c.** $f'(x)$

Representation of Functions by Power Series

Geometric Power Series:

In this section we will look at representing a function by a power series.

Consider $f(x) = \frac{1}{1-x}$ this closely resembles the geometric series

$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, |r| < 1$ if a = 1 and $r = x$ therefore $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Ex: Find the power series for $f(x) = \frac{4}{x+2}$, centered at 0

Ex: Find the power series for $f(x) = \frac{1}{x}$, centered at 1.

Operations with Power Series

Let $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$

1. $f(kx) = \sum a_n k^n x^n$
2. $f(x^N) = \sum a_n x^{nN}$
3. $f(x) \pm g(x) = \sum (a_n \pm b_n) x^n$

Ex: Find the power series centered at 0, for $f(x) = \frac{3x-1}{x^2-1}$

Taylor and Maclaurin Series

The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \Sigma a_n(x - c)^n$ for all x in an open interval I containing c, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

Def'n of Taylor Series and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots$$

is called the **Taylor series for $f(x)$ at c**. Moreover, if $c = 0$, then the series is the **Maclaurin series for f**.

If you know the pattern for the coefficients of the Taylor polynomial for a function, you can extend the pattern easily to form the corresponding Taylor series.

Ex: Use the function $f(x) = \sin x$ to form the Maclaurin series $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ and determine the interval of convergence.

Convergence of a Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Guidelines for Finding a Taylor Series

1. Differentiate $f(x)$ several times and evaluate each derivative at c. Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$ and determine the interval of convergence for the resulting power series.

3. Within the interval of convergence, determine whether or not the series converges to $f(x)$.

Ex: Find the Maclaurin series for $f(x) = \sin x^2$