Functions of Several Variables

Intro to Functions of Several Variables

Every function you have dealt with to this point has been a function of a single variable, f(x). We have to extend the things we learned so far in calculus now to functions of multiple variables. For many equations we need to work with more than one independent variable for example W= Fd.

Notation:

1 independent variable	*2 independent variables*	*3 independent variables*
$y = f(x) = 3x^2 + 2x + 1$	$z = f(x,y) = 5x^2 + 4xy$	$w = f(x,y,z) = x + 2z - 3y^2 z$

Functions of Two Variables:

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x,y) in D there corresponds unique real number f(x,y), then f is called **a function of x and y**. The set of values (x,y) is the **domain** of f and the set of values f(x,y) is the **range**.

x and y are the independent variables and z = f(x,y) is the dependent variable.

to draw a mapping picture of z=f(x,y) we need to think of $\Re^2 \to \Re$

Ex: State the domain and range

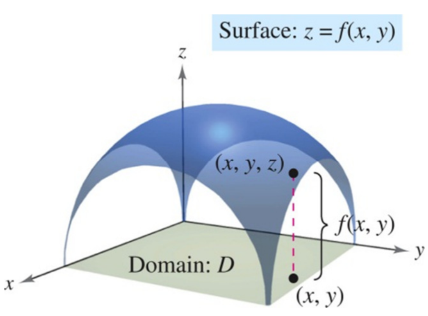
a. $f(x,y) = \sqrt{16 - x^2 - y^2}$

b. $g(x,y) = \ln(x - y^2)$

All the basic operations for functions work the same way (adding, subtracting, etc.)

The Graph of a Function of Two Variables:

The graph of a function of two variables is the set of all points (x,y,z) for which z = f(x,y) where (x,y) is in the domain of f. These graphs will be interpreted as surfaces in space.

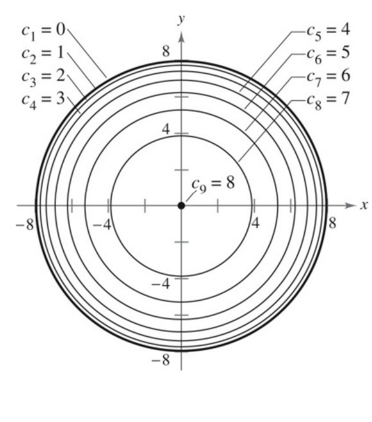


We need to notice the graph of z = f(x,y) is a surface whose projection onto the x – y plane is D.

Ex: Find the domain, range and describe the shape of $f(x,y) = \sqrt{25 - x^2 - y^2}$

Level Curves:

Another way to view a surface for a function of two variables is to observe the scalar field where f(x,y) = c where c is a real number assigned to the point (x,y). The scalar field f(x,y) = c is called a **level curve**.



Functions of More Variables

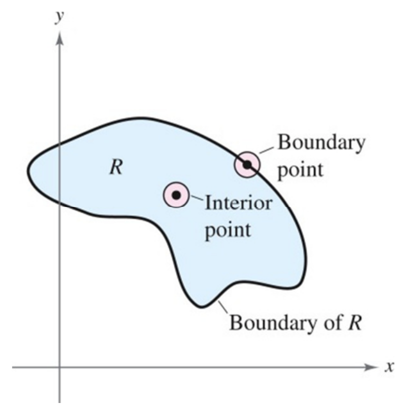
A function of three variables (x, y, z) is a rule that assigns to each ordered triple in the domain exactly on real number w = f(x, y, z).
This is function that maps R^3 to R^1

A way to view a function of three variables is through **level surfaces**.

Limits and Continuity

A **δ – neighborhood** about (x_0, y_0) is any point in the disk centered at (x_0, y_0) where radius of the disk is δ > 0.

$$\delta - neighborhood = \left\{(x_0, y_0 \middle| \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\right\}$$



Any point in the disk is in the δ – neighborhood of the center

An **interior point** of a region R is any point (x,y) such that there exists a δ – neighborhood with center (x,y) such that the entire δ – neighborhood is also inside the region R.

A **boundary point** of a region R is any point (x, y) such that any δ – neighborhood with center (x, y) contains points interior to R and points outside of R.

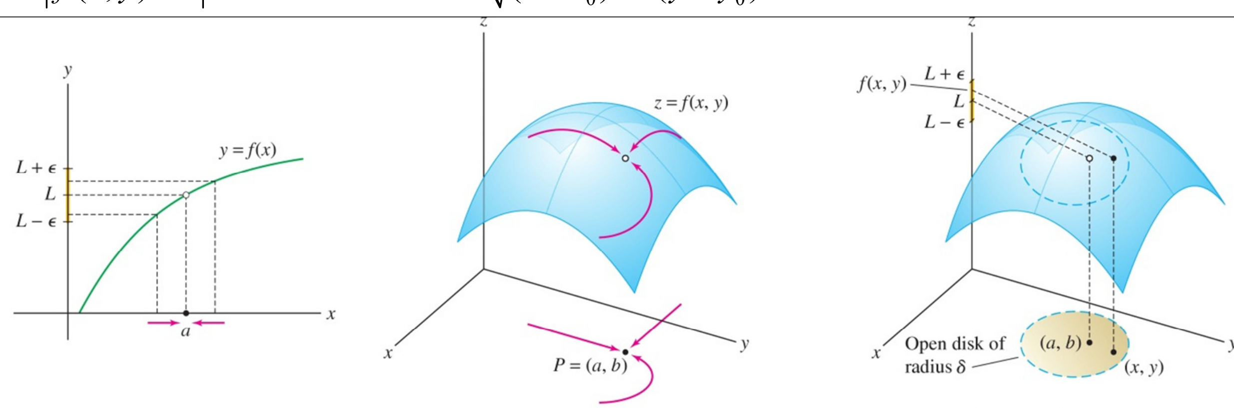
A region R is **open** if it does not contain all of its boundary points and a region R is **closed** if it contains all its boundary points. A region can be neither open nor closed.

Limit of a Function of Two Variables:
Let f be a function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

If for each ε > 0 there exists a δ > 0 such that

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$



(A) In one variable, we can approach a from only two possible directions.

(B) In two variables, (x, y) can approach P = (a, b) along any direction or path.

(C) |f(x, y) − L| < ϵ for all (x, y) inside the disk

FIGURE 2

This implies that for any point (x, y) ≠ (x_0, y_0) in the disk of radius δ, the value f(x,y) lies between L + ε and L – ε.

With functions of a single variable we can approach a value x = c for the left and the right. Now we are approaching a point in the plane and are approaching from infinitely many paths.

To show that a limit does not exist you just need to find two paths of approach for which the limit is not the same.

Ex: Show the limit does not exist $\lim_{(x,y)\to(0,0)} f(x,y)=\dfrac{xy}{x^2+y^2}$

Limits of functions of several variables have the same properties regarding sums, differences, products, and quotients as the limits of single variables.

Ex: Evaluate $\lim_{(x,y)\to(1,2)} f(x,y)=\dfrac{3xy}{x^2+y^2}$

Remember:
Continuity for a function f of a single variable at a point x = c exists only if all three of the following condition hold

1. f(c) is defined
2. $\lim_{x\to c} f(x)$ exists
3. $f(c)=\lim_{x\to c} f(x)$

<u>Continuity of a Function of Several Variables:</u>
A function f of two variables is **continuous at a point** (x_0, y_0) in an open region R if $f(x_0, y_0)$ is equal to the limit of f(x, y) as (x, y) approaches (x_0, y_0).

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0)$$

The function f is **continuous in the open region** R if it is continuous at every point in R

Recall how continuity relates to different functions such as polynomials, rationals, etc.
Ex: Discuss the continuity of the following functions

a. $f(x,y)=4x^5y^3-3x^4y^2+7x^3y+6$
b. $g(x,y)=\dfrac{2x+3y}{x^2-y^2}$
c. $h(x,y)=\dfrac{-3xy}{x^2+y^2}$

<u>Partial Derivatives</u>

How will the value of a function be affected by a change in one of its independent variables? If we look at each independent variable one at a time this can be achieved using partial derivatives.

<u>Partial Derivatives of a Function of Several Variables:</u>
If z = f(x, y), then the first partial derivative of f with respect to a and y are the functions f_x and f_y defined by

$$f_x(x,y)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$f_y(x,y)=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

provided the limits exist

To find each partial derivative we take the derivative with respect to one variable while holding the other variable constant.

Notation for Partial Derivatives:

For z = f(x,y), the partial derviatives f_x and f_y are denoted

$$\frac{\partial}{\partial x} f(x,y) = f_x(x,y) = z_x = \frac{\partial z}{\partial x} \text{ and } \frac{\partial}{\partial y} f(x,y) = f_y(x,y) = z_y = \frac{\partial z}{\partial y}$$

The first partials evaluated at point (a,b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a,b) \text{ and } \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a,b)$$

Ex: Find f_x and f_y for each function

a. $f(x,y) = x^4 y^7$

b. $g(x,y) = 4x^2 \sin(5xy)$

c. $f(x,y) = \tan(x^3 y^4)\ln(x^5 + y^6)$

Partial derivatives for functions of three or more variables would work the same way. Take the derivative with respect to one variable at a time while holding all the other variables constant.

Ex: Find f_x, f_y, and f_z for the given functions

a. $f(x,y,z) = \cos(3x + 4y - 5z)$

b. $w = \ln(x^2 + y^2 + z^2)$

Higher order partial derivatives are just a process of taking the derivative with respect to a given variable multiple times while continuing to hold all other variables constant. We can also change which variable we take the derivative with respect to at each step. These are called mixed partial derivatives

Notation:

Differentiate twice with respect to x

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

Differentiate twice with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Differentiate with respect to x and then with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \quad \text{(mixed partial derivatives)}$$

If f is a functions of x and y, f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x,y) in R, $f_{xy}(x,y) = f_{yx}(x,y)$

Ex: Find f_{xx} f_{yy} f_{xy} and f_{yx} for $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$

Geometric representation of the partial derivatives

If the surface z = f (x,y) is intersected with the plane y = b the z = f(x,b) is the trace curve (curve formed by the intersection). Therefore f_x (a, b) represents the slope of the tangent line at the point (a, b, f(a, b)). The curve and the tangent line will be in the same plane y = b. Similarly, f_y(a, b) represents the slope of the tangent line to the curve given by the intersection of z = f(x, y) with the plane x = a at the point (a, b, f(a, b))

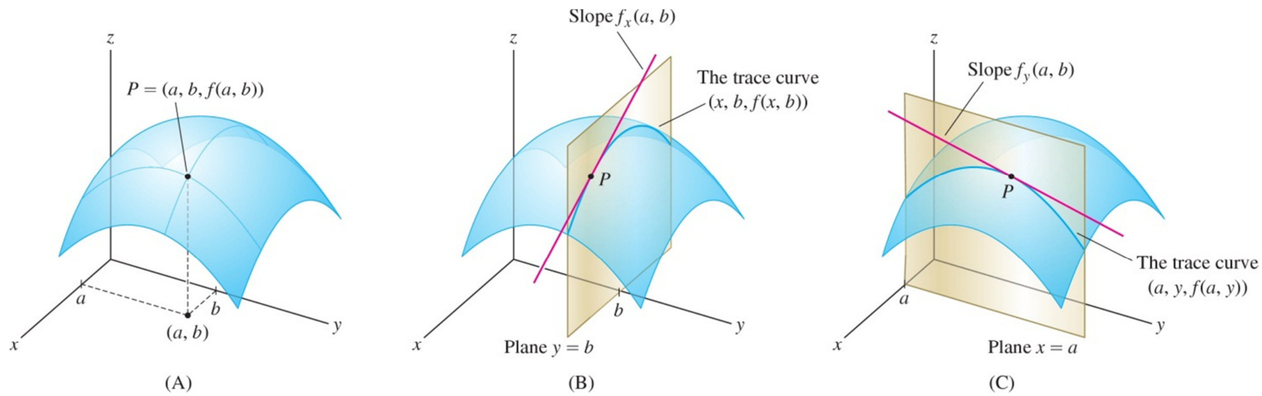


FIGURE 1 The partial derivatives are the slopes of the vertical trace curves.

Ex: The plane x = 2 intersects the surface $f(x, y) = x^4 + 6xy - y^4$ in a certain curve Find the slope of the tangent line to this curve at (2, 2, 24)

Directional Derivatives and Gradients

To determine the slope at a point on a surface, you need a new type of derivative called a directional derivative.

Directional Derivative:

The directional derivative of z = f(x,y) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is given by:

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)u_1 + f_y(x, y)u_2$$

The directional derivative is the slope of the tangent line at point Q to the trace curve obtained when intersecting the graph with a vertical plane through Q in the direction of **u**. (u must be a unit vector)

For three variables it works much the same

Ex: Find $D_{\mathbf{u}} f(x, y)$ in the direction of the vector

a. $f(x, y) = 4x^3 e^{2y}$ in the direction of $\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$

b. $z = \sin 3x - \cos 4y$ in the direction of $\mathbf{u} = \langle 3, -4 \rangle$

c. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ in the direction of $\mathbf{u} = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$

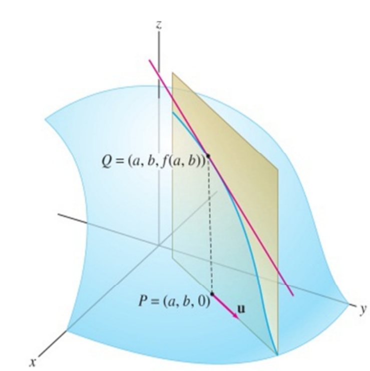


FIGURE 6 $D_{\mathbf{u}} f(a, b)$ is the slope of the tangent line to the trace curve through Q in the vertical plane through P in the direction $\mathbf{u}$.

Reasoning behind the directional derivative formula:

We wish to find the rate of change of z at (x_0, y_0, z_0) in the direction of an arbitrary unit vector **u** = <a,b>. To do this we consider the surface *S* with equation $z = f(x,y)$ and use the point $P(x_0, y_0, z_0)$ that lies on z. The vertical plane that passes through *P* in the direction of **u** intersects *S* in a curve *C*. The slope of the tangent line *T* to *C* at point *P* is the rate of change of z in the direction of **u.**

If $Q(x, y, z)$ is another point on *C* and P' and Q' are the projections of *P*, *Q* onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to **u** and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar *h*. Therefore, $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$ and

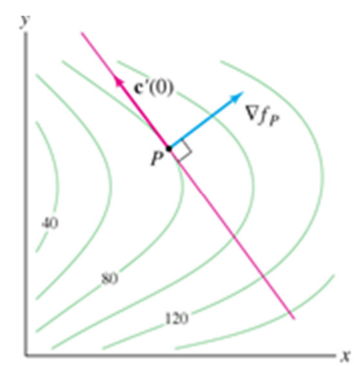
$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

If we take the limit as h goes to zero, we obtain the rate of change of z(with respect to distance) in the direction of **u**.

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$
$$= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

The Gradient of a Function

Let z = f(x,y) be a function of x and y such that f_x and f_y exist. Then we define the **gradient of f** (denoted ∇f) as the vector: $\nabla f = \langle f_x, f_y \rangle$. In three variables $\nabla f = \langle f_x, f_y, f_z \rangle$. At each point P there is a unique direction in which f(x,y) increases most rapidly. That direction is perpendicular to the level curves and is given by the gradient. The gradient at a point P is denoted ∇f_p.



As you can see the directional derivative is really the dot product of the gradient and a unit vector **u**.

Properties of the Directional Derivative and Gradient

$$D_{\mathbf{u}} f(P) = \nabla f_P \bullet \mathbf{u} = \|\nabla f_P\| \cos\theta$$

- ∇f points in the direction of maximum increase of f and point P, or in other words θ = 0 and ∇f and **u** are parallel
- $-\nabla f$ points in the direction of maximum rate of decrease at P or in other words θ = π and ∇f and **u** are parallel but opposite.
- Any direction orthogonal to ∇f is a direction of zero change since θ = π/2 in other words ∇f is normal to the surface f at point P.

Linearization, Tangent Planes, and Normal Planes

The concept of the tangent line in single variable calculus becomes the tangent plane in two variables.

Remember:

If a function f(x) is differentiable at x = a, then

- there exists a tangent line to the graph at x = a with slope $f'(a)$
- there exists a linear approximation to f(x) in an interval containing a

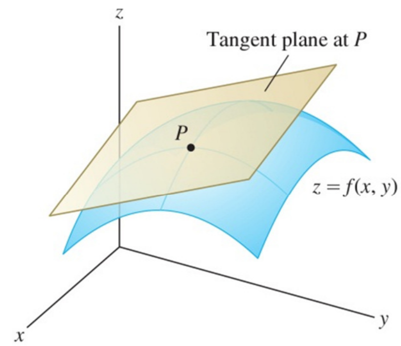


FIGURE 1 Tangent plane to the graph of $z = f(x, y)$.

The second one is what we need for differentiability of functions of two variables

Recall that the linearization of f(x) at x = a is $L(x) = f(a) + f'(a)(x-a)$

A continuous function f(x, y) is differentiable if it is **locally linear**, that is, it its graph looks flatter and flatter as we zoom in on a point P = (a, b, f(a,b)) and eventually becomes indistguishable from the tangent plane.

If a tangent plane to z at P = (a,b, f(a,b)) exists, then its equation must be z = L(x,y), where L(x, y) is the **linearization** at (a, b) defined by

$$z = L(x, y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Ex: Find the linearization of $f(x, y) = x^2y + 2y^3$ at point (2, -1)

Why must this be the tangent plane? Because it is the unique plane that contains the two tangent lines to the two vertical trace curves through P.

A function f(x,y) is differentiable if $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous in a neighborhood of (a,b).

If a function is differentiable at (a, b), then a tangent plane to the surface exists at the point (a, b, f(a,b)).

The line through P orthogonal to the plane is the surfaces **normal line** at P = (a, b, f(a,b)) and is found by $x = a + f_x(a,b)t \quad y = b + f_y(a,b)t$

Ex: Find the equation of the tangent plane to $f(x, y) = \dfrac{x}{\sqrt{y}}$ at (4, 4)

For a surface given by z = f(x,y), you can convert to the general form F(x,y,z) = f(x, y) – z =0 (this will give us the equation of a plane more similar to what we have seen before).

The <u>tangent plane</u> to F(x,y, z) at P = (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

and the <u>normal line</u> is $x = x_0 + f_x(x_0, y_0, z_0)t$, $y = y_0 + f_y(x_0, y_0, z_0)t$, $z = z_0 + f_z(x_0, y_0, z_0)t$

Ex: Find the equation of the tangent plane and normal line to $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$ at (-2, 1, -3)

Chain Rules for Functions of Several Variables

Recall that the chain rule for functions of a single variable gives the rule for differentiating a composition function. If y = f(x) and x = g(t), where f and g are differentiable then y = f(g(t)) and $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = f'(g(t))g'(t)$. The same can be done for a function of two variables z = f(x,y) where x = g(t) and y = h(t) we can write z = f(g(t),h(t)).

Chain Rule for One Independent Variable:

Let z = f(x, y), where f is a differentiable function of x and y. If x = g(t) and y = h(t),where g and h are differentiable functions of t, the z is a differentiable function of t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

This can also be extended to a function of three variables, just continue the pattern.

Ex: Let $z = \sin(x - y)$, where $x = t^2$ and y = 1

Ex: Find $\frac{dw}{dt}$ if $w = \ln(x^2 + y^2 + z^2)$ and $x = 2t$, $y = \frac{1}{3}t^3$, and $z = t^2$

Chain Rule for Two Independent Variable:

Let w = f(x, y), where f is a differentiable function of x and y. If x = g(s,t) and y = h(s,t),such that the first partials all exist, then

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

For functions w = f(x, y, z) and z = k(t) there would just be a third part to theses sums in terms of z.

Ex: Find $\frac{\partial w}{\partial t}$ if $w = y^3 - 3x^2y$ and $x = e^{2s-t}$, $y = 3t^2e^s$

Ex: Find $\frac{\partial w}{\partial t}$ if $w = x^2 + y^2 + z^2$ and $x = t\sin s$, $y = t\cos s$, and $z = st^2$

The chain rule can be used to give a more complete description of the process of implicit differentiation learned back in calc I.

Implicit Differentiation for Functions of a Single Variable:

If the function F(x,y) = 0 defines y implicitly as a function of x, then

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}, \quad F_y(x,y) \neq 0$$

This method of implicit differentiation tends to be easier than the methods used in calc 1.

Ex: Find $\frac{dy}{dx}$ for $y^3 + y^2 - 5y - x^2 + 4 = 0$

Implicit Differentiation for Functions of Multiple Variables:

If the function F(x,y,z) = 0 defines z implicitly as a function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}, \quad F_z(x,y,z) \neq 0$$

Ex: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $x^3y^2z^4 + 2x^2y^5z^3 - 3x^4 + 2y^5 - 8z^2 = -9$

Extrema of Functions of Two Variables and Optimization

Let f(x,y) be defined on a region R containing the point (a,b) then

- f(a, b) is a **local maximum** of f if f(a, b) ≥ f(x, y) for all (x,y) in an open disk centered at (a,b)
- f(a, b) is a **local minimum** of f if f(a, b) ≤ f(x, y) for all (x,y) in an open disk centered at (a,b)

A point (a,b) is a **critical point** of f(x,y) if $f_x(a,b) = f_y(a,b) = 0$ or if one does not exist.

Ex: Find the critical points for $f(x,y) = y^2x - yx^2 + xy$

First Derivative Test (Fermat's Theorem):

If f(x, y) has a local maximum or minimum at (a,b), then (a,b) is a critical point of f(x, y).

Remember not all critical points give rise to extrema. Any critical point that is not an extreme value is saddle point.

Ex: Find all extrema for $f(x,y) = 2x^2 + y^2 + 8x - 6y + 20$

A value that helps in determining extrema in the **discriminant** or **Hessian** and is found by $D = f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$ *(use determinants to help remember)*

Second Derivative Test:

Let (a,b) be a critical point of f(x, y) and f_{xx}, f_{yy}, and f_{xy} are continuous near (a,b) then,

- f(a, b) is a local maximum if $f_{xx} < 0$ and D > 0
- f(a, b) is a local minimum if $f_{xx} > 0$ and D > 0
- f(a, b) is a is a saddle point if D < 0
- the test is inconclusive if D = 0

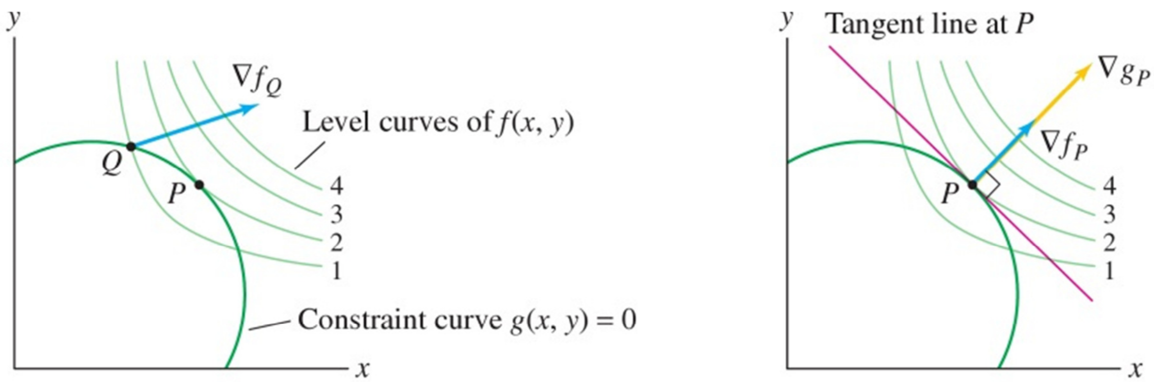
Ex: Find all extrema and saddle points for $f(x,y) = x^3 - xy + y^3$

Ex: Find all extrema and saddle points for $f(x,y) = x^4 + y^4 - 4xy + 1$

Lagrange Multipliers

Sometimes we need to find the extreme values of a function whose domain is constrained to lie within some particular subset of the plane. The method is called the **Lagrange Multipliers**.

Suppose we are trying to find the extreme values of f(x,y) subject to the constraint g(x,y) = 0. This means we want to increase or decrease f(x,y) while remaining on some curve. The gradient ∇f points in the direction of maximum increase or decrease, but we can't move in the gradient direction because it would take us off the curve.
So if the gradient points toward the right we move in that direction along the curve until the ∇f becomes orthogonal to the constraint curve g(x,y) = 0 or in other words, ∇f is parallel to ∇g. If we solve the equation $\nabla f(x,y) = \lambda \nabla g(x,y)$, where λ is known as the **Lagrange Multiplier,** we will find extreme values due to the constraint.



(A) f increases as we move to the right along the constraint curve.

(B) The local maximum of f on the constraint curve occurs where ∇f_P and ∇g_P are parallel.

FIGURE 2

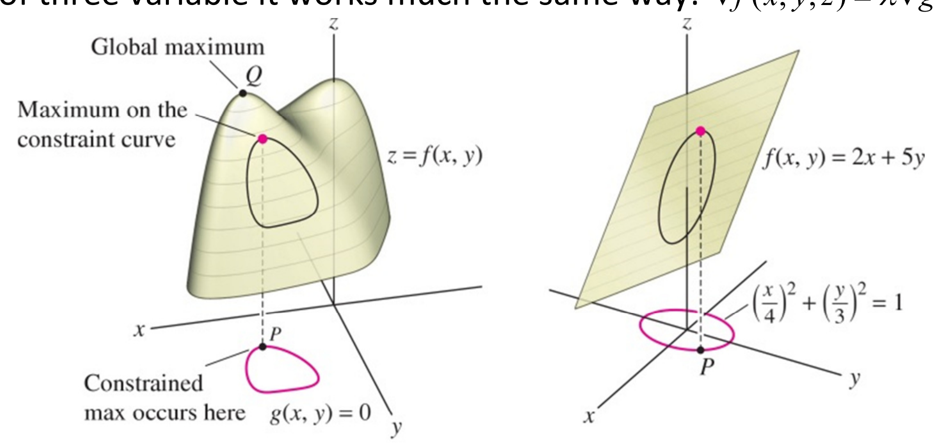
<u>**Lagrange Multipliers:**</u>

Assume that f(x,y) and g(x,y) are differentiable functions. If f(x,y) has a local minimum of a local maximum on a constraint curve g(x,y) = 0 at P = (a,b), and if $\nabla g_P \neq 0$, then there is a scalar λ such that

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

A point P = (a,b) satisfying the Lagrange equations is called a **critical point** for the optimization problem with constraint and f(a,b) is called a **critical value**.

For functions of three variable it works much the same way: $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$



(A) (B)

FIGURE 6

To optimize f(x,y) subject to the constraint g(x,y) = 0 generally we can do the following

1. **Solve the three equations in three unknowns**

   $f_x(x,y) = \lambda g_x(x,y)$

   $f_y(x,y) = \lambda g_y(x,y)$

   $g(x,y) = 0$

   In most cases you will:

   - Solve for λ in terms of x and y and set those equations equal.
   - Solve for x and y using the constraint equation.

2. Evaluate f at all the critical points found in the previous step. The largest value is the maximum and the smallest is the minimum.

Ex: Find the extreme values of $f(x,y) = xy$ subject to $\frac{x^2}{8} + \frac{y^2}{2} = 1$

Ex: Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the cylinder $x^2 + y^2 = 1$

Ex: Find the extreme values of $f(x,y,z) = 2x^2 + y^2 + 3z^2$ subject to $2x - 3y - 4z = 49$