Introduction to Differential Equations

Definitions and Terminology

Differential Equation: An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE).**

Classification by Type: If an equation contains only ordinary derivatives of one or more dependent variables with the respect to a single independent variable is said to be an **ordinary differential equation (ODE)**.

Ex: $\frac{dy}{dx}+5y=e^x, \frac{d^2y}{dx^2}-\frac{dy}{dx}+6y=0$ and $\frac{dx}{dt}+\frac{dy}{dt}=2x+y$

An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation (PDE)**.

Ex: $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}-2\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial t}, \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

Most text will write ordinary derivatives using either **Leibniz notation**:

$\frac{dy}{dx},\frac{d^2y}{dx^2},\frac{d^3y}{dx^3}...$ or **prime notation:** $y',y'',y'''....$

Using prime notation the previous ODE's can be written as:

$$y'+5y=e^x \text{ and } y''-y'+6y=0$$

Remember the notation for 4th, 5th, etc. derivatives

Other notations:

Newton's dot notation: $\frac{d^2s}{dt^2}=-32$ becomes $\ddot{s}=-32$

Subscript notation: the previous second PDE becomes:

$$u_{xx}=u_{tt}-2u_t$$

Classification by Order:

The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation.

Ex: $\frac{d^2y}{dx^2}+5\left(\frac{dy}{dx}\right)^3-4y=e^x$ is a second order

We can express an *n*th ordinary differential equation in one dependent variable by general form:

$$F(x,y,y',y'',...y^{(n)})=0$$

where **F** is a real valued function of *n + 2* variables. We can solve this equation for the highest derivative by:

$$\frac{d^ny}{dx^n}=f(x,y,y',...y^{(n-1)})$$

where *f* is a real valued continuous function and is referred to as the **normal form** of a differential equation.

Classification by Linearity:
An nth-order ordinary differential equation is said to be linear if F is linear in $y, y', y'', y''', y^{(n)}$. This means that an nth order **ODE** is linear when

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

The two important cases are:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \text{ and } a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Characteristic Two Properties of a Linear ODE

- The dependent variable y and all its derivatives $y', y'', y'''...$ are of the first degree, that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, ..., a_n$, of $y, y', y''...$ depend at most on the independent variable x.

Ex: $(y - x)dx + 4xdy = 0$, $y'' - 2y' + y = 0$, and $\frac{d^3 y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$

A **nonlinear** ODE is one that is not linear.

Solutions of an ODE: any function ϕ , defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth order ODE reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

Occasionally it will be convenient to denote the solution by an alternate symbol $y(x)$

Interval of Definition:
The interval I in the previous definition is called the **interval of definition**, the **interval of existence**, the **interval of validity**, or the **domain of the solution** and can be open closed or infinite.

Ex: Verify that the indicated function is a solution of a given differential equation on $(-\infty,\infty)$.

a. $dy/dx = xy^{1/2}; y = (1/16)x^4$ **b.** $y'' - 2y' + y = 0; y = xe^x$

A solution in which the dependent variable is expressed solely in term of the independent variable and constants is said to be an **explicit solution**. The previous solutions are explicit

Implicit solution of an ODE:
A relation $G(x,y) = 0$ is said to be an **implicit solution** of an ODE on an interval I, provided there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

Ex: Verification of an Implicit Solution
The relation $x^2 + y^2 = 25$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

On the interval -5 < *x* < 5. By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dy}y^2 = \frac{d}{dx}25 \text{ or } 2x + 2y\frac{dy}{dx} = 0$$

Families of Solutions:
When solving 1*st* order DE's $F(x,y,y') = 0$, we usually obtain a solution containing a single arbitrary constant or parameter *c*.

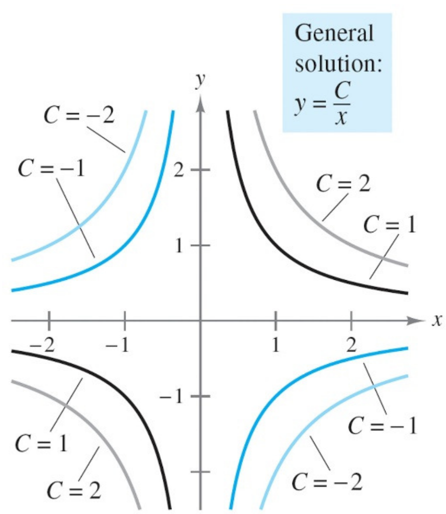
A solution containing an arbitrary constant represents a set $G(x,y,c) = 0$ of solutions called a **one parameter family of solutions**.

When solving an *n*th order DE we seek an ***n*th parameter family of solutions**.

A solution of a differential equation that is free of arbitrary parameters is called a **particular solution.**
Ex: $y = cx - x\cos x$ is a solution to $xy' - y = x^2$ (1 parameter)
Ex: $y = c_1e^x + c_2xe^x$ *is a solution to* $y'' - 2y' + y = 0$ (2 parameter)

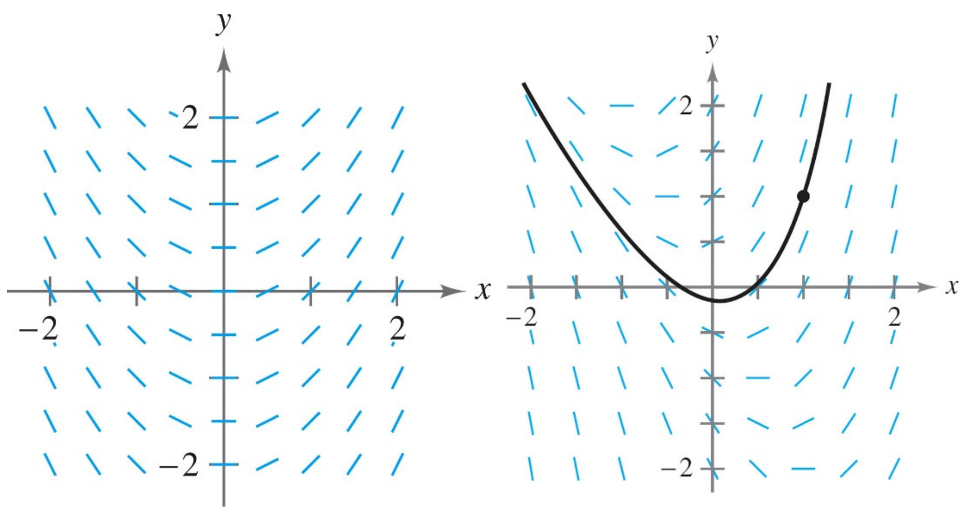
Geometrically, the general solution of a first order differential equation represents a family of curves know as solution curves, one for each assigned value of the arbitrary constant.

If $y = \frac{C}{x}$ is the general solution to $xy' + y = 0$ the solution curves would look like the following:



Solving differential equations analytically can be difficult or even impossible. However, there is a graphical approach you can use to learn a lot about the solution of a differential equation.

Consider a differential equation of the form $y' = F(x, y)$ where $F(x,y)$ is some expression in x and y. At each point (x,y) in the x-y plane where F is defined, the differential equation determines the slope $y = F(x, y)$ of the solution at that point. If you draw short line segments with slope $F(x,y)$ at selected points (x,y) in the domain of F, then these line segments for a **slope field**.



A slope field shows the general shape of all the solutions and can be helpful in getting a visual perspective of the solutions to a differential equation.

<u>Initial Value Problems</u>

We are often interested in problems in which we seek a solution $y(x)$ of a DE so that $y(x)$ satisfies prescribed conditions. This is called an **initial-valued problem** (**IVP**). The values of $y(x)$ at a single point (x_0, y_0) are called **initial conditions**.

First Order IVP's

Solve: $\dfrac{dy}{dx} = f(x, y)$

Subject to: $y(x_0) = y_0$

Second Order IVP's

Solve: $\dfrac{d^2y}{dx^2} = f(x, y, y')$

Subject to: $y(x_0) = y_0$, $y'(x_0) = y_1$

Ex: If $y = ce^x$ is one parameter family of solutions of the simple 1st order **DE** $y' = y$. All solutions are defined on the interval $(-\infty,\infty)$. If we impose an initial condition $y(0) = 3$, we can find a solution to the IVP.

If we place the initial condition $y(1) = -2$ we can find another solution.

2 major questions:

- Does a solution to the problem exist?
- If the solution exists, is it unique?

Picard's existence of a unique solution:
Let R be a rectangular region in the x-y plane defined by $a \le x \le b,\ c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists an interval I centered at x_0 and a unique function $y(x)$ defined on I satisfying the IVP.

EX: If $x = c_1 \cos 4t + c_2 \sin 4t$ is a two parameter family of solutions of $x'' + 16x = 0$ **.** Find a solution of the IVP:

$$x'' + 16x = 0,\ x\left(\frac{\pi}{2}\right) = -2,\ x'\left(\frac{\pi}{2}\right) = 1$$

Separable Variables:
At this point you should have a strong background in integration of single and multi-variable functions.

Solution by Integration:
Consider the first order differential equation

$$\frac{dy}{dx} = f(x,y)$$

When f is a single variable function of x then the differential equation becomes

$$\frac{dy}{dx} = g(x)$$

and can be solved by integration. If $g(x)$ is a continuous function, then integrating both sides gives us

$$\int dy = \int g(x)dx$$
$$y = G(x) + C$$

where G(x) is an antiderivative of $g(x)$.

Ex: Solve the following first order separable DE

a. $\frac{dy}{dx} = 1 + xe^{2x}$ **b.** $\frac{dy}{dx} = \sec x$

Separable Equation:
A first order differential equation of the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

is said to be **separable** or to have **separable variables**.

Separable: $\frac{dy}{dx} = y^2 x e^{3x+4y}$ **Non-separable:** $\frac{dy}{dx} = y + \sin x$

Separable equations can be written as

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

$$or$$

$$h(y)dy = g(x)dx$$

we then integrate both sides to solve

$$\int h(y)dy = \int g(x)dx$$

$$H(y) = G(y) + C$$

If we follow this procedure to solve separable variables a one-parameter family of solutions is obtained.

Ex. Solve the following first order DE:

a. $(1+x)dy - ydx = 0$ **b.** $\frac{dy}{dx} = -\frac{x}{y}, y(4) = 3$

c. $xe^{-y} \sin x dx - y dy = 0$ **d.** $xy^4 dx + (y^2+2)e^{-3x} dy = 0$

- Unless convenient leave in implicit form, however the interval over which the solution exists in not easy to see.
- Make sure when separating the variables, the divisors are not zero.

Ex: Solve: $\frac{dy}{dx} = y^2 - 4, \; y(0) = 2$

Ex: Solve: $(e^{2y} - y)\cos x \frac{dy}{dx} = e^y \sin 2x, y(0) = 0$

Homogeneous Equations:

Definition: If a function has the property that

$$f(tx, ty) = t^n f(x, y)$$

For some real number n, then f is said to be a **homogeneous function of degree n**.

Ex: Determine which of the following are homogeneous

a. $f(x, y) = x^2 - 3xy + 5y^2$ **b.** $f(x, y) = \sqrt[3]{x^2 + y^2}$

c. $f(x, y) = x^3 + y^3 + 1$ **d.** $f(x, y) = \frac{x}{2y} + 4$

If $f(x,y)$ is a homogeneous function of degree n, notice that we can write

$$f(x, y) = x^n f\left(1, \frac{y}{x}\right) \; and \; f(x, y) = y^n f\left(\frac{x}{y}, 1\right)$$

where $f(1, y/x)$ and $f(x/y, 1)$ are both homogeneous of degree n.

Ex: **a.** $f(x, y) = 6xy^3 - x^2 y^2$ **b.** $f(x, y) = x^2 - y$ **c.** $f(x, y) = x^2 + 3xy + y^2$

Definition: A differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be **homogeneous** if both coefficients M and N are homogeneous functions of the same degree.

Method of Solution:

A homogeneous D.E. can be solved by using either substitution $y = ux$ or $x = vy$, where u and v are new dependent variables to reduce the homogeneous D.E. into a separable first order D.E.

$$y = ux, \quad dy = udx + xdu$$

$$M(x, y)dx + N(x, y)dy = 0$$

$$M(x,ux)dx + N(x,ux)(udx + xdu) = 0$$

Since homogeneous

$$x^n M(1,u)dx + x^n N(1,u)(udx + xdu) = 0$$

$$[M(1,u) + uN(1,u)]dx + xN(1,u)du = 0$$

Which gives $$\frac{dx}{x} + \frac{N(1,u)du}{M(1,u) + uN(1,u)} = 0$$

Ex: Solve the following:

a. $(x^2 + y^2)dx + (x^2 - xy)dy = 0$

b. $(2\sqrt{xy} - y)dx - xdy = 0$

c. $2x^3 ydx + (x^4 + y^4)dy = 0$

d. $x\frac{dy}{dx} = y + xe^{y/x}$, $y(1) = 1$

Exact Equations:

Differential of a Function of Two Variables:

If $z = f(x, y)$ is a function of two variables with continuous first partial derivatives in a region R of the xy - plane, then its differential is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad (1)$$

In the special case when $f(x, y) = c$, where c is a constant, then (1) implies

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \qquad (2)$$

In other words, given the one parameter family of function $f(x, y) = c$, we can generate a first order DE by computing the differential of both sides of the equality. For example, if $x^2 - 5xy + y^3 = c$, then (2) gives the first order DE

$$(2x - 5y)dx + (-5x + 3y^2)dy = 0 \text{ or } \frac{dy}{dx} = \frac{5y - 2x}{-5x + 3y^2} \qquad (3)$$

Exact Equation: A differential expression

$$M(x,y)dx + N(x,y)dy$$

is an **exact differential** in a region **R** of the xy - plane if it corresponds to the differential of some function $f(x,y)$ defined in **R**. A first order DE of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be an **exact differential equation** if the expression on the left hand side is an exact differential.

Ex: $x^2y^3dx + x^3y^2dy = 0$ is an exact differential equation, because its left-hand side is an exact differential:

$$d\left(1/3\,x^3y^3\right) = x^2y^3dx + x^3y^2dy$$

The Criterion for an Exact Differential:

Let $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region **R** defined by $a<x<b$ **,** $c<y<d$**.** Then a necessary and sufficient condition that

$$M(x,y)dx + N(x,y)dy$$

be an exact differential if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (4)$$

Method of Solution:

Given an equation in the differential form
$M(x,y)dx + N(x,y)dy = 0$, determine whether the equality in (4) holds. If it does, then there exists a function *f* for which

$$\frac{\partial f}{\partial x} = M(x,y)$$

We can find *f* by integrating $M(x,y)$ with respect to *x* while holding *y* constant:

$$f(x,y) = \int M(x,y)dx + g(y) \qquad (5)$$

where the arbitrary function $g(y)$ is the "constant" of integration. Now differentiate (5) with respect to *y* and assume $\partial f/\partial y = N(x,y)$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y) = N(x,y)$$

This gives

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx) \qquad (6)$$

Finally integrate (6) with respect to *y* and substitute the result in (5). The implicit solution of the equation is

$$f(x,y) = c$$

It is important to realize the expression $N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx)$ is independent of x, because

$$\frac{\partial}{\partial x}\left[N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx)\right]=\frac{\partial N}{\partial x}-\frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\int M(x,y)dx\right)=\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=0$$

Ex: Solve the following

- **a.** $2xydx + \left(x^2 - 1\right)dy = 0$
- **b.** $\left(e^{2y} - y\cos xy\right)dx + \left(2xe^{2y} - x\cos xy + 2y\right)dy = 0$
- **c.** $\frac{dy}{dx}=\frac{xy^2-\cos x\sin x}{y(1-x^2)},\ y(0)=2$

Integrating Factors:

It is sometimes possible to convert a non-exact differential equation into an exact equation by multiplying by some function $\mu(x, y)$ called an **integrating factor**.

However, the resulting exact equation

$$\mu M(x,y)dx + \mu N(x,y)dy=0$$

May not be equivalent to the original in the sense that a solution of one is also the solution of the other. It is possible for a solution to be lost or gained as a result of the multiplication.

Ex: Solve $(x+y)dx+x\ln x dy=0$ using $\mu(x,y)=\frac{1}{x}$, on $(0,\infty)$.

The hard part is finding μ, one way is to use partial DE, we can't do that yet. Consider this, by the criterion $M(x,y)dx + N(x,y)dy = 0$ will be exact iff $(\mu M)_y=(\mu N)_x$.

Using the product rule we get

$$\mu_x N - \mu_y M = \left(M_y - N_x\right)\mu$$

M,N,M_y,N_x are known functions of x and y. So another way is to assume u is a function of one variable. In this case we get the two scenarios:

$$M(x,y)dx + N(x,y)dy = 0$$

- If $\left(M_y-N_x\right)/N$ is a function of x alone, then an integrating factor is

$$\mu(x)=e^{\int\frac{M_y-N_x}{N}dx}$$

- If $\left(N_x - M_y\right)/M$ is a function of y alone, then an integrating factor is

$$\mu(y)=e^{\int\frac{N_x-M_y}{M}dy}$$

Ex: $xydx + \left(2x^2+3y-20\right)dy = 0$ (this is not exact)

Linear Equations:

A first order DE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \quad (1)$$

is said to be a **linear equation** in the dependent variable *y*.

Standard Form:

By dividing both sides of the previous equation by the lead coefficient we obtain the **standard form**:

$$\frac{dy}{dx} + P(x)y = f(x) \qquad (2)$$

We seek a solution, on *I* for which both coefficient functions ***P*** and ***f*** are **continuous**.

Integrating factor: with differentials we can write the standard form as

$$dy + \left[P(x)y - f(x)\right]dx \;=\; 0$$

With linear equation you can have the property that you can always find a factor $\mu(x)$ such that

$$\mu(x)dy + \mu(x)\left[P(x)y - f(x)\right]dx = 0 \quad (3)$$

is an exact differential equation.
If it is exact then

$$\frac{\partial}{\partial x}\mu(x) = \frac{\partial}{\partial y}\mu(x)\left[P(x) - f(x)\right]$$

or

$$\frac{d\mu}{dx} = \mu P(x)$$

Looking a little closer this is a separable equation that we can use to find $\mu(x)$

$$\frac{d\mu}{\mu} = P(x)dx$$

$$\ln|\mu| = \int P(x)dx$$

$$\mu(x) = e^{\int P(x)dx}$$

We don't use a constant of integration because (3) is unaffected by a constant multiple.
If we use this integrating factor in (3) we get

$$e^{\int P(x)dx}dy + e^{\int P(x)dx}P(x)ydx = e^{\int P(x)dx}f(x)dx$$

rewrite as

$$d\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x)dx$$

then integrate

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}f(x)dx + c$$

or

$$y = e^{-\int P(x)dx}\int e^{\int P(x)dx} f(x)dx + ce^{-\int P(x)dx}$$

<u>Solving a Linear First Order Equation:</u>

1. Put a linear equation into the standard form

2. From the standard form identify *P(x)* and then find the integrating factor $\mu(x) = e^{\int P(x)dx}$

3. Multiply the standard form of the equation by the integrating factor, the left hand side of the resulting equation is automatically the derivative of the integrating factor and *y*.

$$\frac{d}{dx}\left[e^{\int P(x)dx}\right]y = e^{\int P(x)dx} f(x)$$

4. Integrate both sides of this last equation.

Ex: Solve the following

a. $\frac{dy}{dx} - 3y = 0$

b. $x\frac{dy}{dx} - 4y = x^6 e^x$

c. $(x^2 - 9)\frac{dy}{dx} + xy = 0$

d. $\frac{dy}{dx} + 2xy = x,\ y(0) = -3$

e. $x\frac{dy}{dx} + y = 2x,\ y(1) = 0$

f. $\frac{dy}{dx} = \frac{1}{x + y^2},\ y(-2) = 0$

<u>Equations of Bernoulli</u>

Here we do not look at a particular type of DE but instead a specific equations that can be turned into DE we already know how to solve.

Bernoulli's Equation:

$$\frac{dy}{dx} + P(x)y = f(x)y^n$$

where *n* is any real number, notice that for $n = 0$ and $n = 1$ the equation is linear. (We already know how to solve that and we wouldn't use the proceeding method)

If $y \neq 0$ we can write the equation as

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = f(x)$$

Let $w = y^{1-n},\ n \neq 0,\ n \neq 1$ then

$\frac{dw}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ (use implicit diff where *y* is a function of *x*) substitute this into the altered Bernoulli equation to produce a linear equation of the form:

$$y^{-n}\left(y^n \frac{dw}{dx}\frac{1}{(1-n)}\right) + P(x)w = f(x)$$

$$\frac{dw}{dx}+(1-n)P(x)w=(1-n)f(x)$$

solve this for w then use $w=y^{1-n}$ you get a solution to the Bernoulli equation.

Ex: Solve the following

a. $\frac{dy}{dx}+\frac{1}{x}y=xy^2$

b. $\frac{dy}{dx}+y=e^x y^2$

c. $x\frac{dy}{dx}-(1+x)y=xy^2$

Ricatti's Equation: the nonlinear differential equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2$$

if y_1 is a known particular solution of the Ricatti Equation then the substitutions

$$y = y_1 + u \text{ and } \frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

into the Ricatti Equation leads to

$$\frac{dy_1}{dx} - P(x) - Q(x)(y_1 + u) + R(x)(y_1 + u)^2$$

then since

$$\frac{dy_1}{dx} - P(x) - Q(x)y_1 - R(x)y_1^2 = 0$$

$$\frac{du}{dx} - (q + 2y_1R)u = Ru^2$$

this is actually now a Bernoulli equation with n = 2, it can then be reduced to the linear equation

$$\frac{dw}{dx} + (q + 2y_1R)w = -R$$

by substituting $w = u^{-1}$

Ex: Solve $\frac{dy}{dx} = 2 - 2xy + y^2$