Exponential and Logarithmic Functions

Exponential Functions and Their Graphs:

Section Objectives: Students will know how to recognize, graph, and evaluate exponential functions.

The **exponential function** ***f(x)*** **with base *b*** is denoted by

$$f(x) = b^x$$

where ***b* > 0**, ***b*** $\neq$ **1**, and ***x*** is any real number.

The exponential function is called a transcendental function.

Graphs of Exponential Functions

For the equation $f(x) = b^x$

1. The **domain** is **$(-\infty, \infty)$,** so its continuous for all real numbers
2. The **range** is **$(0, \infty)$**
3. The ***y*-intercept** is **(0, 1)**
4. ***y* = 0** is a **horizontal asymptote** *(only on one end)*
5. ***f (x)*** is **increasing** if ***b* > 1**
6. ***f (x)*** is **decreasing** if **0 < *b* < 1**
7. ***f* (x)** is **one to one**

Ex: Graph the following exponential functions.

a. $f(x) = 2^x$ **b.** $f(x) = 2^{-x} = (1/2)^x$

Ex: Graph each of the following on the same coordinate axes.

a. $f(x) = 3^x$ b. $g(x) = 3^{x+1}$

c. $h(x) = 3^x - 2$ d. $k(x) = -3^x$

Ex: See what happens if we change the base.

a. $g(x) = 4^x$ b. $f(x) = 4^{-x+2}$ c. $h(x) = 4^{-x+2} - 3$

Exponential Functions Properties

1. Exponent laws:

$$a^x a^y = a^{x+y} \quad \left(a^x\right)^y = a^{xy} \quad \left(ab\right)^x = a^x b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \quad \frac{a^x}{b^x} = a^{x-y}$$

2. $a^x = a^y$ iff $x = y$

3. For $x \neq 0$, $a^x = b^x$ iff $a = b$

The Natural Base e

e is an irrational number, where **e = 2.718281828459…**

The exponential function with base e is called the **natural exponential function**

$$f(x) = e^x$$

It can be shown that f **(x) = (1 + 1/x)$^{\mathbf{x}}$** $\rightarrow$ **e as x** $\rightarrow \infty$ **.**

x	**$[1 + \left(\frac{1}{x}\right)]^x$**
1	2
2	2.25
4	2.44141
12	2.61304
365	2.71457
8,760	2.71813
525,600	2.71828

Ex: Draw the graph of **f(x) = e$^{\mathbf{x-2}}$**

Applications

Let **A** be the amount in the account after **n** pay periods, let **P** be the principal, and let **x** be the periodic interest rate. Then the compounded interest after one year can be calculated by using, **A = P(1 + x)$^{\mathbf{n}}$**

The compound interest formula after t years is

$$\mathbf{A = P(1 + r/n)^{nt}}$$

A is the amount in the account after t years.
P is the principal.
r is the annual interest rate.
n is the number of pay periods per year.

An investment of $5,000 is made into an account that pays 6% annually for 10 years. Find the amount in the account if the interest is compounded:

a. **annually (n = 1)** *($8954.24)* b. **quarterly (n = 4)** *($9070.09)*
c. **monthly (n = 12)** *($9096.98)* d. **daily (n = 365)** *($9110.14)*

As n increases, so does A, but the rate of increase slows. **What would happen if n** $\rightarrow \infty$ **?**

Working with $\mathbf{A = P(1 + r/n)^{nt}}$ let n/r = x or r/n = 1/x we get

$$A = P\left[\left(1+\frac{1}{x}\right)^{x}\right]^{rt}$$

then as n→∞, x also goes to ∞, and the following continuously compounded interest formula can be derived.

$$\boldsymbol{A = Pe^{rt}}$$

A is the amount in the account after t years.
P is the principal.
r is the annual interest rate.

Ex: Continue the example previous with the interest compounded continuously.

Logarithmic Functions:

Section Objectives: Students will know how to recognize, graph, and evaluate logarithmic functions, rewrite logarithmic functions with a different base, use properties of logarithms to evaluate, rewrite, expand, or condense logarithmic expressions.

Since the exponential function $\mathbf{f(x) = b^x}$ is one-to-one, its inverse is a function. The function given by

$$f(x) = \log_b x$$

where $\boldsymbol{x > 0}$, $\boldsymbol{b > 0}$, and $\boldsymbol{b \neq 1}$ is called the **logarithmic function** with base *b.*

Conversely, the logarithmic function with base *b* is the inverse of the exponential function with base *b*; thus

$$\boldsymbol{y = \log_b x \text{ if and only if } x = b^y}$$

(the two statements are equivalent)

Ex: Evaluate each of the following.

a. $\mathbf{f(x) = \log_2 32}$
b. $\mathbf{f(x) = \log_3 1}$
c. $\mathbf{f(x) = \log_4 2}$
d. $\mathbf{f(x) = \log_{10}(1/100)}$

We call the logarithmic function with base 10 the **common logarithmic function**. This is the function that corresponds to the LOG button on our calculators. The common logarithmic function is the one function for which we do not write the base.

Graphs of Logarithmic Functions

The **graph of the logarithmic function** comes directly from the properties of the graph of the exponential function and the inverse relationship. Inverse functions graphs are symmetric around the y = x line.

Look at $f(x) = 2^x \quad and \quad g(x) = \log_2 x$

For $f(x) = \log_b x$

- The **domain** is **(0, ∞).**
- The **range** is **($-\infty, \infty$)**.
- The ***x*-intercept** is **(1, 0)**.
- The ***y*-axis** is a **vertical asymptote**.
- The function is **increasing (*b* > 1)** and **decreasing (0 < b < 1)**
- **Continuous** over its entire domain

Ex: Sketch the graph of the following.
a) $y = \log_{10} x$ b) $y = \log_{10}(x + 2)$ c) $y = \log_{10}(x + 2) - 1$

The Natural Logarithmic Function

The logarithmic function with base **e** is called the **natural logarithmic function** and is denoted by:

$$f(x) = \log_e x = \ln x, \quad x > 0$$

$f(x) = \ln x$ is the inverse of $g(x) = e^x$

Draw the graph of f(x) = lnx
Ex: Find the domain of the following and graph
a. $f(x) = \ln(x-2)$ b. $g(x) = \ln(2-x)$

Properties of Log functions	Properties of Natural Logs
1. $\log_b 1 = 0$ because $b^0=1$	1. $\ln 1 = 0$ because $e^0=1$
2. $\log_b b = 1$ because $b^1 = b$	2. $\ln e = 1$ because $e^1=e$
3. $\log_b b^x = x$ and $a^{\log_a x} = x$	3. $\ln e^x=x$ and $e^{\ln x} = x$
4. If $\log_b x = \log_b y$, then $x = y$.	4. If $\ln x = \ln y$, then $x = y$

You can see that these are the same since lnx is just a log base e

Ex: Simplify

a. $\log_4 4 =$ b. $\log_5 5^x =$ c. $6^{\log_6 20} =$

Ex: Simplify

a. $\ln\frac{1}{e}$ b. $e^{\ln 5}$ c. $\frac{\ln 1}{3}$ d. $2\ln e$

<u>More properties of Logarithms</u>

Since ***y* = log*b*x if and only if *x* = b*y***, then the following **properties of logarithms** are similar to the exponent properties.

Let ***b*** be a positive real number such that ***b*** $\neq$ **1**, let ***n*** be a real number, and let ***u*** and ***v*** be positive real numbers.

$\log_b(uv) = \log_b u + \log_b v$	$\ln(uv) = \ln u + \ln v$
$\log_b(u/v) = \log_b u - \log_b v$	$\ln(u/v) = \ln u - \ln v$
$\log_b u^n = n \log_b u$	$\ln u^n = n \ln u$

The use of these properties is usually known as the expansion of logs

Ex: Expand the logarithmic expression. Write each log as ln 2 or ln 3

a) ln 6 **b)** ln(2/27)

Ex: Verify that $-\log_{10}\frac{1}{100} = \log_{10} 100$

Ex: Expand Logarithms completely

a. $\log_4 5x^3 y$ **b.** $\ln\left(\frac{\sqrt{3x-5}}{7}\right)$

Ex: Rewrite as a single logarithm

a. $\frac{1}{2}\log x + 3\log(x+1)$ **b.** $\frac{1}{3}\left[\log_2 x + \log_2(x+1)\right]$

Change of Base

Our calculators have only two buttons for logarithmic functions, base 10 and base *e*. If we want to evaluate logs with other bases we need this formula.

The Change-of-Base Formula

Let ***a***, ***b***, and ***x*** be positive real numbers such that $a \neq 1$ and $b \neq 1$.

Base b	Base 10	Base e
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

Use calculator to solve $\log_4 30$

Exponential and Logarithmic Equations:

Section Objectives: Students will know how to solve exponential and logarithmic equations.

2 Ways To Solve An Exponential or Log Equation

1. Use one to one property
2. Use inverse property

One-to-One Properties	***Inverse Properties***
1. $a^x = a^y$ if and only if $x = y$.	**1.** $\log_a a^x = x$
2. $\log_a x = \log_a y$ if and only if $x = y$.	**2.** $a^{\log_a x} = x$

Ex: Solve

a. $2^x = 32$ **b.** $\ln x - \ln 3 = 0$ **c.** $\left(\frac{1}{3}\right)^x = 9$

d. $e^x = 7$ **e.** $\ln x = -3$ **f.** $\log_{10} x = -1$

Strategies for Solving Exponential and Log Equations

1. Isolate the exponential or Log
2. Rewrite the original equation in a form that allows you to use the one-to-one property.
3. Rewrite the exponential equation in Log form and apply the Inverse Property
4. Rewrite Log equation in exponential form and apply the Inverse Property.

Solve:

a. $4^x = 72$ **b.** $3(2^x) = 42$ **c.** $e^x + 5 = 60$ **d.** $2(3^{2t-5}) - 4 = 11$

Solving Exponentials of Quadratic Type:

Solve: $e^{2x} - 3e^{x} + 2 = 0$

Exponentiating Both Sides of the Equation

Solve: $\ln x = 3$ $\ln x = 2$ $\log_3(5x-1) = \log_3(x+7)$

$5 + 2\ln x = 4$ $2 \log_5 3x = 4$

Extraneous Solutions

$\log_{10} 5x + \log_{10}(x-1) = 2$

Check the answers!!!

Applications

How long would it take for an investment of $500 to double if the interest were compounded continuously at 6.75%?