Diff. Eq. Test #3 Problem Set **Prof. G. Buthusiem**

1. Solve $x^2y'' - xy' + y = 2x$

2. Find two power series solutions of the given DE about the ordinary point x = 0:

$(x^2+1)y'' - 6y = 0$

3. Find two series solutions about the singular point x = 0 for $2xy'' + 5y' + xy = 0$

4. Solve $6x^2y'' + 5xy' - y = 0$

5. Find two power series solutions about the point x = 0 for $y'' + xy = 0$

6. Find the Laplace transform for the following

a. $f(t) = t^5$ **b.** $f(t) = (2t-1)^3$ **c.** $f(t) = \cos 5t + \sin 2t$

7. Find the inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\}$

8. Use the Laplace transform to solve the IVP: $\frac{dy}{dt} + 2y = t, \ y(0) = -1$

Diff EQ Test #3 Problem Set KEY G. Buthusiem

1. $x^2y'' - xy' + y = 2x$ 1st Solve $x^2y'' - xy' + y = 0$

$a=1 \quad b=-1 \quad c=1$ Aux. eq. $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \quad m = 1$ repeated

$y_c = c_1 x + c_2 x \ln x$

$y_1 = x \quad y_2 = x \ln x$

Now Solve

$x^2y'' - xy' + y = 2x \longrightarrow y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{2}{x}$

$f(x) = 2/x \qquad y_p = u_1 y_1 + u_2 y_2$

$u_1' = \frac{-y_2 f(x)}{w} = \frac{-x\ln x(2/x)}{w} = -\frac{2\ln x}{x}$

$u_2' = \frac{y_1 f(x)}{w} = \frac{x \cdot 2/x}{w} = \frac{2}{x}$

$W = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x(\ln x + 1) - x\ln x$
$= x$

$u_1 = \int u_1' = \int -\frac{2\ln x}{x}\,dx = -(\ln x)^2$

$u_2 = \int u_2' = \int \frac{2}{x}\,dx = 2\ln|x|$

$y_p = -x(\ln x)^2 + 2x(\ln x)^2 = x\ln x$

$y = y_c + y_p = c_1 x + c_2 x\ln x + x\ln x$

2. $(x^2+1)y'' - 6y = 0$

$y = \sum_{n=0} c_n x^n \quad y' = \sum_{n=1} c_n n x^{n-1} \quad y'' = \sum_{n=2} c_n n(n-1) x^{n-2}$

$(x^2+1)\sum_{n=2} c_n n(n-1)x^{n-2} - 6\sum_{n=0} c_n x^n = 0$

$\sum_{n=2} c_n n(n-1)x^n + \sum_{n=2} c_n n(n-1) x^{n-2} - \sum_{n=0} 6c_n x^n = 0$

$k=n \qquad k=n-2,\ n=k+2 \qquad k=n$

$\sum_{k=2} c_k k(k-1)x^k + \sum_{k=0} c_{k+2}(k+2)(k+1)x^k - \sum_{k=0} 6c_k x^k = 0$

$c_2(2)x^0 + c_3(3)(2)x^1 - 6c_0x^0 - 6c_1x^1 + \sum_{k=2}\left((k^2-k-6)c_k + (k+2)(k+1)c_{k+2}\right)x^k = 0$

$(2c_2 - 6c_0) + (6c_3 - 6c_1)x^1 + \sum \cdots$

$2c_2 - 6c_0 = 0 \quad 6c_3 - 6c_1 = 0 \quad c_{k+2} = \dfrac{-c_k(k-3)\cancel{(k+2)}}{\cancel{(k+2)}(k+1)}$

$c_2 = 3c_0 \qquad c_3 = c_1$

$k=2 \quad c_4 = \dfrac{-c_2(-1)}{3} = \dfrac{3c_0}{3} = c_0 \qquad k=6 \quad c_8 = \dfrac{-c_6(3)}{7} = \dfrac{c_0(3)}{5\cdot 7}$

$k=3 \quad c_5 = \dfrac{-c_3(0)}{4} = 0$

$k=4 \quad c_6 = \dfrac{-c_4(1)}{5} = -\dfrac{c_0}{5}$

$k=5 \quad c_7 = -\dfrac{c_5(2)}{6} = 0$

if $y = \sum_{n=0} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$

$= c_0 + c_1x + 3c_0x^2 + c_1x^3 + c_0x^4 + 0x^5 - \dfrac{c_0}{5}x^6 + \cdots$

$= c_1(x + x^3) + c_0\left(1 + 3x^2 + x^4 - \tfrac{1}{5}x^6 + \cdots\right)$

$\uparrow y_1 \qquad \uparrow y_2$

3. $2xy'' + 5y' + xy = 0$

$$y = \sum_{n=0} c_n x^{n+r} \quad y' = \sum_{n=0} c_n (n+r) x^{n+r-1} \quad y'' = \sum_{n=0} c_n (n+r)(n+r-1) x^{n+r-2}$$

$$2x \sum_{n=0} c_n (n+r)(n+r-1) x^{n+r-2} + 5 \sum_{n=0} c_n (n+r) x^{n+r-1} + x \sum_{n=0} c_n x^{n+r} = 0$$

$$\sum_{n=0} 2c_n (n+r)(n+r-1) x^{n+r-1} + \sum 5c_n (n+r) x^{n+r-1} + \sum c_n x^{n+r+1} = 0$$

$$\sum_{n=0} c_n (n+r)(2n+2r+3) x^{n+r-1} + \sum_{n=0} c_n x^{n+r+1} = 0$$

$$x^r \left(\sum_{n=0} c_n (n+r)(2n+2r+3) x^{n-1} + \sum_{n=0} c_n x^{n+1} \right) = 0$$

$k = n-1$ $\quad$ $k = n+1$

$n = k+1$ $\quad$ $n = k-1$

$$x^r \left(\sum_{k=-1} c_{k+1} (k+r+1)(2k+2r+5) x^k + \sum_{k=1} c_{k-1} x^k \right) = 0$$

$$x^r \left(c_0 (r)(2r+3) x^{-1} + c_1 (r+1)(2r+5) x^0 + \sum_{k=1} \left[c_{k+1} (k+r+1)(2k+2r+5) + c_{k-1} \right] x^k \right) = 0$$

$r(2r+3) = 0$ $\quad$ if $(r+1)(2r+5)c_1 = 0$

$r = 0 \quad r = -3/2$ $\quad$ $\therefore c_1 = 0$

$$c_{k+1} = \frac{-c_{k-1}}{(k+r+1)(2k+2r+5)}$$

if $r = 0$

$$c_{k+1} = \frac{-c_{k-1}}{(k+1)(2k+5)}$$

$k=1$ $\quad$ $k=2$ $\quad$ $k=3$

$$c_2 = \frac{-c_0}{2(7)} \qquad c_3 = \frac{-c_1}{3(9)} = 0 \qquad c_4 = \frac{-c_2}{4(11)} = \frac{c_0}{2 \cdot 4 \cdot 7 \cdot 11}$$

1st Solution

$$y = \sum_{n=0} c_n x^{n+r} = \sum_{n=0} c_n x^{n+0} = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$= c_0 + 0x - \frac{c_0}{2 \cdot 7} x^2 + 0x^3 + \frac{c_0}{2 \cdot 4 \cdot 7 \cdot 11} x^4 + \cdots$$

$$= c_0 \left(1 - \frac{1}{2 \cdot 7} x^2 + \frac{1}{2 \cdot 4 \cdot 7 \cdot 11} x^4 - \cdots \right)$$

#3 continued

$r = -3/2$

$$C_{k+1} = \frac{-C_{k-1}}{(k+1/2)(2k+2)} = \frac{-C_{k-1}}{(2k-1)(k+1)}$$

$k=1$ $\quad$ $k=2$ $\quad$ $k=3$

$$C_2 = \frac{-C_0}{1(2)} \qquad C_3 = \frac{-C_1}{3(3)} = 0 \qquad C_4 = \frac{-C_2}{5(4)} = \frac{C_0}{2\cdot4\cdot5}$$

$$y = \sum_{n=0} C_n x^{n+r} = x^{-3/2} \sum C_n x^n = x^{-3/2}(C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots)$$

$$= x^{-3/2}\left(C_0 + 0x - \frac{C_0}{2}x^2 + 0x^3 + \frac{C_0}{2\cdot4\cdot5}x^4 + \cdots\right)$$

2nd Solution, $\quad y = C_0 x^{-3/2}\left(1 - \frac{1}{2}x^2 + \frac{1}{2\cdot4\cdot5}x^4 - \cdots\right)$

#4 $\quad 6x^2y'' + 5xy' - y = 0$

$a=6 \quad b=5 \quad c=-1$

Aux. eq. $\quad 6m^2 - m - 1 = 0$

$(3m+1)(2m-1) = 0$

$m = -1/3 \quad m = 1/2$

$$\boxed{y = C_1 x^{-1/3} + C_2 x^{1/2}}$$

See #5 on next page

#6. a. $\mathcal{L}\{t^5\} = \frac{5!}{s^6}$ $\quad$ b. $\mathcal{L}\{(2t-1)^3\} = \mathcal{L}\{8t^3 - 12t^2 + 6t - 1\}$

$$= 8\cdot\frac{3!}{s^4} - 12\frac{2!}{s^3} + 6\cdot\frac{1}{s^2} - \frac{1}{s}$$

c. $\mathcal{L}\{\cos 5t + \sin 2t\} = \frac{s}{s^2+25} + \frac{2}{s^2+4}$

5. $y'' + xy = 0$ $\quad x=0$ is an ordinary point since $a_2(x) \neq 0$

$y = \sum_{n=0} c_n x^n \quad y' = \sum_{n=1} c_n n x^{n-1} \quad y'' = \sum_{n=2} c_n (n)(n-1) x^{n-2}$

$$\sum_{n=2} c_n n(n-1)x^{n-2} + x \sum_{n=0} c_n x^n = 0$$

$$\sum_{n=2} c_n n(n-1)x^{n-2} + \sum_{n=0} c_n x^{n+1} = 0$$

$k = n-2$, $n = k+2$ $\qquad$ $k = n+1$, $n = k-1$

$$\sum_{k=0} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1} c_{k-1} x^k = 0$$

$$c_2(2)(1)x^0 + \sum_{k=1} \left(c_{k+2}(k+2)(k+1) + c_{k-1}\right)x^k = 0$$

$2c_2 = 0$ $\qquad c_{k+2} = \dfrac{-c_{k-1}}{(k+2)(k+1)}$

$c_2 = 0$

$k=1 \quad c_3 = \dfrac{-c_0}{3(2)}, \quad k=2 \quad c_4 = \dfrac{-c_1}{4(3)}, \quad k=3 \quad c_5 = \dfrac{-c_2}{5(4)} = 0$

$k=4 \quad c_6 = \dfrac{-c_3}{6\cdot 5} = \dfrac{c_0}{2\cdot 3\cdot 6\cdot 5}, \quad k=5 \quad c_7 = \dfrac{-c_4}{7(6)} = \dfrac{c_1}{3\cdot 4\cdot 6\cdot 7}$

$$y = \sum c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

$$= c_0 + c_1 x + 0x^2 - \frac{c_0}{2\cdot 3}x^3 - \frac{c_1}{3\cdot 4}x^4 + 0x^5 + \frac{c_0}{2\cdot 3\cdot 6\cdot 5}x^6$$

$$= c_0\Big(\underbrace{1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 6\cdot 5}x^6 - \ldots}_{y_1}\Big) + c_1\Big(\underbrace{x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^6 - \ldots}_{y_2}\Big)$$

7. $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s}\right\}$

$$\frac{s+1}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$$

$s+1 = A(s+4) + B(s)$

$s=0$ $\quad s=-4$

$1 = A(4)$ $\quad -3 = -4B$

$A = 1/4$ $\quad B = 3/4$

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s} + \frac{3}{4}\cdot\frac{1}{s+4}\right\} = \frac{1}{4} + 3/4\, e^{-4t}$$

8. $\frac{dy}{dt} + 2y = t \qquad y(0) = -1$

$\mathcal{L}\{\frac{dy}{dt}\} + 2\mathcal{L}\{y\} = \mathcal{L}\{t\}$

$sY(s) - y(0) + 2Y(s) = \frac{1}{s^2}$

$(s+2)Y(s) + 1 = \frac{1}{s^2} \rightarrow Y(s) = \dfrac{\frac{1}{s^2} - 1}{s+2} = \dfrac{1-s^2}{s^2(s+2)}$

$$\frac{1-s^2}{s^2(s+2)} = \frac{A}{s+2} + \frac{B}{s} + \frac{C}{s^2}$$

$1 - s^2 = As^2 + Bs(s+2) + C(s+2)$

$s=0$ $\quad s=-2$ $\quad s=1,\ A=-3/4\ \ C=+1/2$

$1 = C(2)$ $\quad -3 = A(4)$ $\quad 0 = -3/4(1) + B(3) + \frac{1}{2}(3)$

$C = 1/2$ $\quad A = -3/4$ $\quad -3/4 = 3B \rightarrow B = -1/4$

$$Y(s) = -3/4\cdot\frac{1}{s+2} - \frac{1}{4}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s^2}$$

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left\{-\frac{3}{4}\cdot\frac{1}{s+2} - \frac{1}{4}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{1}{s^2}\right\}$$

$$\boxed{y(t) = -\frac{3}{4}\cdot e^{-2t} - \frac{1}{4} + \frac{1}{2}t}$$