Calc III Test #3 Problem Set G. Buthusiem

1. Evaluate the double integral by changing to polar coordinates

$$\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\,dx\,dy$$

2. Evaluate $\int_0^{\pi/2} \int_0^{y/2} \int_0^{1/y} \sin y\,dz\,dx\,dy$

3. Find the volume of the solid bounded by $z = 9 - x^2$, $z=0$, ~~$y=0$~~, $y=2x$
& $y = 2x$, $x \geq 0$

4. Convert to either cylindrical or spherical coordinates to evaluate the integral $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\,dz\,dy\,dx$

5. Find the Jacobian for $x = uv - 2u$, $y = uv$

6. Evaluate $\int_C 3(x-y)\,ds$ along the path $C: \mathbf{r}(t) = t\mathbf{i} + (2-t)\mathbf{j}, 0 \leq t \leq 2$

7. Determine whether the vector field is conservative
$\mathbf{F}(x,y) = 15x^2y^2\,\mathbf{i} + 10x^3y\,\mathbf{j}$, if so find the potential function.

8. Find the gradient vector field for $f(x,y,z) = \sqrt{x^2+4y^2+z^2}$

9. Use Green Theorem to evaluate $\int_C y^2\,dx + xy\,dy$

C: boundary of a given region lying between the graphs of

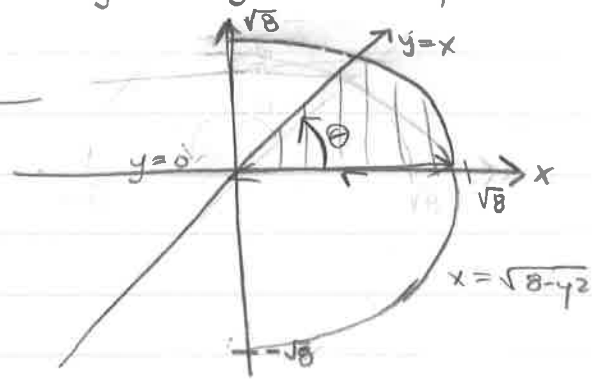
$y=0$, $y=\sqrt{x}$, $x=9$

Calc 3 Test #3 Problem Set KEY G. Buthusiem

1. $\int_0^2 \int_y^{\sqrt{8-y^2}} \sqrt{x^2+y^2}\, dx\, dy$

Interval:

$0 \le y \le 2$, $y < x \le \sqrt{8-y^2}$



$f(x,y) = \sqrt{x^2+y^2} \longrightarrow \sqrt{r^2} = r$

$0 \le \theta \le \pi/4 \longleftarrow$

$0 \le r \le \sqrt{8}$

$$= \int_0^{\pi/4} \int_0^{\sqrt{8}} r \cdot r\, dr\, d\theta$$

$$= \int_0^{\sqrt{8}} r^2\, dr \cdot \int_0^{\pi/4} d\theta = \frac{r^3}{3}\Big|_0^{\sqrt{8}} \cdot \pi/4 = \frac{(\sqrt{8})^3}{3}\left(\frac{\pi}{4}\right)$$

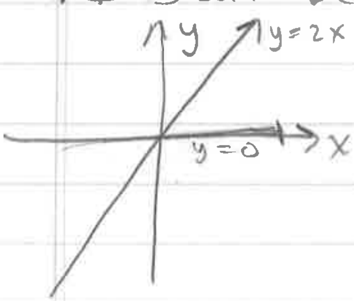
$$= \frac{8\sqrt{8}\,\pi}{12} = \frac{2\sqrt{8}\,\pi}{3}$$

2. $\int_0^{\pi/2} \int_0^{y/2} \int_0^{\frac{1}{y}} \sin y\, dz\, dx\, dy = \int_0^{\pi/2} \int_0^{y/2} z \sin y \Big|_0^{1/y} dx\, dy$

$$= \int_0^{\pi/2} \int_0^{y/2} \frac{\sin y}{y}\, dx\, dy = \int_0^{\pi/2} x\left(\frac{\sin y}{y}\right)\Big|_0^{y/2} dy = \int_0^{\pi/2} \frac{\sin y}{2}\, dy$$

$$= \frac{-1}{2}\cos y \Big|_0^{\pi/2} = -\frac{1}{2}\left(\cos \pi/2 - \cos(0)\right) = -\frac{1}{2}(0-1) = \boxed{\frac{1}{2}}$$

3. It should be clear that $0 \le z \le 9-x^2$ then look at x & y



so $0 \le y \le 2x$ but what about x

we can see $x \ge 0$, & from the equation

$z = 9-x^2$ x max's out at 3 so

$0 \le x \le 3$

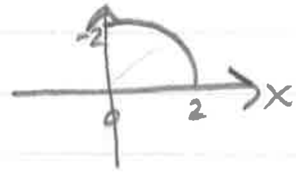
$$\therefore V = \int_0^3 \int_0^{2x} \int_0^{9-x^2} dz\, dy\, dz = \int_0^3 \int_0^{2x} z \Big|_0^{9-x^2} dy\, dx$$

$$= \int_0^3 \int_0^{2x} (9-x^2)\, dy\, dx = \int_0^3 y(9-x^2)\Big|_0^{2x} dx = \int_0^3 2x(9-x^2)\, dx$$

$$= 9x^2 - \frac{1}{2}x^4 \Big|_0^3 = 9(3)^2 - \frac{1}{2}(3)^4 = 81/2$$

4. $$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\; dz\,dy\,dx$$

$0 \le x \le 2 \quad 0 \le y \le \sqrt{4-x^2} \quad 0 \le z \le \sqrt{16-x^2-y^2}$



z is a upper hemisphere radius 4

x & y is a quarter circle radius 2

Convert to cylindrical coordinates

$$0 \le z \le \sqrt{16-(r\cos\theta)^2-(r\sin\theta)^2}$$
$$0 \le z \le \sqrt{16-r^2}$$

$0 \le r \le 2$
$0 \le \theta \le \pi/2$

$$= \int_0^{\pi/2}\int_0^2\int_0^{\sqrt{16-r^2}} r\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^2 zr^2\Big|_0^{\sqrt{16-r^2}} dr\,d\theta = \int_0^{\pi}\int_0 \left(\sqrt{16-r^2}\right) r^2\,dr\,d\theta$$

for $\int_0^2 \sqrt{16-r^2}\; r^2\,dr$ use trig sub.

if $r = 4\sin\beta$ then $\sqrt{16-r^2} = 4\cos\beta$

$dr = 4\cos\beta\,d\beta$

$$= \int_0^2 16\cos^2\beta\, 16\sin^2\beta\, d\beta = 16^2\int_0^2 \left(\frac{1+\cos 2\beta}{2}\right)\left(\frac{1-\cos 2\beta}{2}\right) d\beta$$

$$= \frac{16^2}{4}\int_0^2 (1-\cos^2 2\beta)\,d\beta = \frac{16^2}{4}\int_0^2 \left(1-\left(\frac{1+\cos 4\beta}{2}\right)\right)d\beta$$

$$= \frac{16^2}{4}\int_0^2\left(\frac{1}{2}-\frac{\cos 4\beta}{2}\right)d\beta = \frac{16^2}{4}\left(\frac{1}{2}\beta - \frac{\sin 4\beta}{8}\right) = 8(4\beta - \sin 4\beta)\Big|_0^2$$

$$\sin 4\beta = 4(\sin\beta\cos\beta)(\cos^2\beta - \sin^2\beta)$$

Rather than switch back to r & evaluate btw $r=0$ & $r=2$ we can stay in β & convert $[0,2]$ to $[0,\pi/3]$ using $r = 4\sin\beta$

$$= 8\left(4\arcsin(r/4) - \frac{4}{1}\left(\frac{r}{4}\cdot\frac{\sqrt{16-r^2}}{4}\right)\left(\frac{16-r^2}{16} - \frac{r^2}{16}\right)\right)\Big|_0^2$$

$$\ldots = \int_0^{\pi/2} 8\left(4\arcsin(1/2) - \frac{4}{16^2}\left(2\sqrt{16-4}\right)\left(16-2(2)^2\right)\right)d\theta$$

$$= \frac{8\pi}{2}\left(4\cdot\frac{\pi}{6} - \frac{4}{16^2}(2\sqrt{12})(8)\right) = \frac{8\pi}{2}\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= 4\pi\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

5. $x = uv - 2w \qquad y = uv$

$\frac{\partial x}{\partial u} = v - 2 \qquad \frac{\partial x}{\partial v} = u$

$\frac{\partial y}{\partial u} = v \qquad \frac{\partial y}{\partial v} = u$

$$\text{Jacobian} = \begin{vmatrix} v-2 & u \\ v & u \end{vmatrix} = u(v-2) - uv = uv - 2u - uv$$
$$= -2w$$

6. $\int_C 3(x-y)\,ds \qquad r(t) = \langle t, 2-t \rangle \qquad 0 \le t \le 2$

$r'(t) = \langle 1, -1 \rangle$

$$\int_0^2 (3(t-(2-t))\sqrt{1^2+1^2}\,dt$$

$$= \int_0^2 3(2t-2)\sqrt{2}\,dt = \int_0^2 6\sqrt{2}(t-1)\,dt = 6\sqrt{2}\left(\frac{t^2}{2} - t\right)\Big|_0^2$$

$$= 6\sqrt{2}\left(4/2 - 2\right) = 0$$

7. $F(x,y) = \langle 15x^2y^2, 10x^3y \rangle \qquad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \rightarrow 30x^2y = 30x^2y$

Its conservative $\therefore \int 15x^2y^2\,dx = \int 10x^3y\,dy =$ potential f.

$$\frac{15x^3}{3}y^2 + g(y) = \frac{10x^3y^2}{2} + h(x)$$

$$5x^3y^2 + g(y) = 5x^3y^2 + h(x) = f(x,y)$$

then $g(y) = h(x)$

$\therefore g(y) = h(x) = C$

So

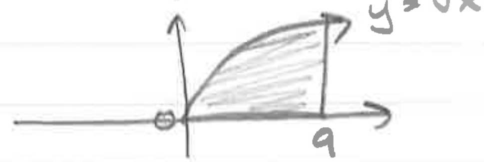
$$f(x,y) = 5x^3y^2 + C$$

8. $f(x,y,z) = \sqrt{x^2+4y^2+z^2}$

$$F(x,y,z) = \left\langle \frac{x}{\sqrt{x^2+4y^2+z^2}}, \frac{4y}{\sqrt{x^2+4y^2+z^2}}, \frac{z}{\sqrt{x^2+4y^2+z^2}} \right\rangle$$

9. $\int_C y^2\,dx + xy\,dy \quad C: y=0,\ y=\sqrt{x},\ x=9$

Region



$M = y^2 \qquad N = xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = y$

$$= \int_0^9 \int_0^{\sqrt{x}} (y-2y)\,dy\,dx = \int_0^9 \int_0^{\sqrt{x}} (-y)\,dy\,dx$$

$$= \int_0^9 -\frac{y^2}{2}\Big|_0^{\sqrt{x}} dx = \int_0^9 -\left(\frac{x}{2}\right) dx = -\frac{x^2}{4}\Big|_0^9$$

$$= -\frac{81}{4}$$