Calc 3 Test #2 Problem Set G. Buthusiem

1. Show that the $\lim\limits_{(x,y)\to(0,0)} \dfrac{-2x^2y^2}{x^4+y^4}$ does not exist.

2. Find $\dfrac{\partial z}{\partial x}$ implicitly for $3x^2 - zx^3 + 5yx = \cos(x^2z+y)$

3. Find $\dfrac{dz}{dt}$ for $z = 5x^2y - \tan(xy+y^2)$ where $x = 3t^2+1$ & $y = \cos t$

4. Given $f(x,y) = -2xy + 3x^2 - 8y^5$ and $\vec{u} = \langle 3,-4\rangle$ find $D_u f(x,y)$ at point $(1,0,3)$. Also find the derivative in the direction of maximum increase at $(1,0,3)$.

5. Find the equation of the tangent plane & normal line to $5zx - 3x^2y + 8y^3 - 9x^3 = 6$ at $(1,0,3)$

6. Find all relative extrema & saddle points for $f(x,y) = x^2 - 3xy - y^2$

7. Find the maximum & minimum values of $f(x,y) = x^2+y^2$ subject to $xy = 1$

8. Evaluate the following double integrals

   a. $\displaystyle\iint_R y\sin(xy)\,dA$ where $R = [1,2]\times[0,\pi]$

   b. $\displaystyle\iint_D x^2+y^2\,dA$ where $D = \{(x,y)\mid 0\le x\le 2,\ x^2<y<2x\}$

Calc 3 Test #2 Problem Set Key G. Bothusien

1. $\lim\limits_{(x,y)\to(0,0)} \dfrac{-2x^2y^2}{x^4+y^4}$

Can't plug (0,0) in so →

Approach from the x-axis $\therefore y=0$

$\lim\limits_{(x,0)\to(0,0)} \dfrac{-2x^2y^2}{x^4+y^4} = \dfrac{-2x^2(0)}{x^4+0} = \dfrac{0}{x^4} = 0$

Approach from the y-axis $\therefore x=0$

$\lim\limits_{(0,y)\to(0,0)} \dfrac{-2x^2y^2}{x^4+y^4} = \dfrac{-2(0)y^2}{0^4+y^4} = \dfrac{0}{y^4} = 0$

Approach from the line $y=x$

$\lim\limits_{(x,x)\to(0,0)} \dfrac{-2x^2y^2}{x^4+y^4} = \dfrac{-2x^2(x^2)}{x^4+x^4} = \dfrac{-2x^4}{2x^4} = -1$

not equal $\therefore \lim\limits_{(x,y)\to(0,0)}$ DNE

2. $3x^2 - zx^3 + 5yx - \cos(x^2z+y) = 0$

$F_x = 6x - 3zx^2 + 5y + \sin(x^2z+y)(2xz)$

$F_z = -x^3 + \sin(x^2z+y)\cdot(x^2)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{3zx^2 - 6x - 5y - 2xz\sin(x^2z+y)}{x^2\sin(x^2z+y) - x^3}$$

3. $z = 5x^2y - \tan(xy+y^2)$ $\quad x = 3t^2+1$ $\quad y = \cos t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$= \left(10xy - [\sec^2(xy+y^2)](y)\right)(6t) + \left(5x^2 - (\sec^2(xy+y^2))(x+2y)\right)(-\sin t)$

$= \left(60t(3t^2+1)\cos t - 6t(\cos t)\sec^2((3t^2+1)\cos t + \cos^2 t)\right) + \longrightarrow$

$\longrightarrow \left(-5(3t^2+1)^2\sin t\right) + \sin t\,(3t^2+1+2\cos t)\sec^2((3t^2+1)\cos t + \cos^2 t))$

4. $\vec{u} = \langle 3,-4\rangle$ $\quad \|\vec{u}\| = \sqrt{9+16} = \sqrt{25} = 5$ not mag $=1$

$\vec{u}_0 = \langle 3/5, -4/5\rangle$

$f(x,y) = -2xy + 3x^2 - 8y^5$

$f_x = -2y + 6x$ $\qquad f_y = -2x - 40y^4$

4. continued

$$D_u f(x,y) = f_x u_1 + f_y u_2$$
$$= (-2y+6x)(3/5) + (-2x-40y^5)(-4/5)$$
$$D_u f(1,0) = (-2(0)+6(1))(3/5) + (-2(1)-40(0)^5)(-4/5)$$
$$= \frac{18}{5} + \frac{8}{5} = \boxed{\frac{26}{5}}$$

∇f points in the direction of maximum increase

$$\nabla f = \langle f_x, f_y \rangle = \langle -2y+6x, -2x-40y^5 \rangle$$

$$\nabla f \text{ at } (1,0,3) = \langle -2(0)+6(1), -2(1)-40(0)^5 \rangle$$
$$= \langle 6, -2 \rangle$$

$$\|\nabla f\| = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10} \neq 1$$

$$\nabla f_0 \text{ at } (1,0,3) = \left\langle \frac{6}{2\sqrt{10}}, \frac{-2}{2\sqrt{10}} \right\rangle = \left\langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \right\rangle$$

$$D_{\nabla f_0} f(1,0) = 6\left(\frac{3}{\sqrt{10}}\right) + (-2)\left(\frac{-1}{\sqrt{10}}\right) = \frac{18+2}{\sqrt{10}} = \boxed{\frac{20}{\sqrt{10}}}$$

5. $5zx - 3x^2y + 8y^3 - 9x^3 = 6$

$F_x = 5z - 6xy - 27x^2 \qquad F_x(1,0,3) = 15 - 27 = -12$

$F_y = -3x^2 + 24y^2 \qquad F_y(1,0,3) = -3$

$F_z = 5x \qquad F_z(1,0,3) = 5$

normal line: $x = x_1 + F_x t \qquad y = y_1 + F_y t \qquad z = z_1 + F_z t$

$x = 1 - 12t \qquad y = -3t \qquad z = 3 + 5t$

tangent plane: $F_x(x - x_1) + F_y(y - y_1) + F_z(z - z_1) = 0$

$$-12(x-1) - 3(y-0) + 5(z-3) = 0$$

$$-12x - 3y + 5z - 3 = 0$$

6. $f(x,y) = x^2 - 3xy - y^2$

$f_x(x,y) = 2x - 3y$
$\rightarrow$
$f_y(x,y) = -3x - 2y$

critical values

$2x - 3y = 0$
$-3x - 2y = 0$
$\rightarrow$
$6x - 9y = 0$
$-6x - 4y = 0$

$-13y = 0$

$\boxed{x = 0 \leftarrow y = 0}$

$f_{xx}(x,y) = 2 \qquad f_{xy}(x,y) = -3$
$f_{yy}(x,y) = -2$

$D = 2(-2) - (-3)^2 = -13$

$f_{xx}(0,0) = 2 \rightarrow$ pos $\qquad D(0,0) = -13 \rightarrow$ neg

$\therefore (0, 0, f(0,0))$ is a saddle point.

7. $f(x,y) = x^2 + y^2 \qquad xy = 1$

$f_x = 2x \qquad g_x = y$
$f_y = 2y \qquad g_y = x$
$\rightarrow \nabla f = \lambda \nabla g \rightarrow$
$2x = \lambda y$
$2y = \lambda x$
$\rightarrow$
$\lambda = \frac{2x}{y}$
$\lambda = \frac{2y}{x}$

$\therefore \lambda = \frac{2x}{y} = \frac{2y}{x} \rightarrow 2x^2 = 2y^2 \Rightarrow x = \pm y$

Plug into $xy = 1 \rightarrow$ if $x = y \rightarrow y(y) = 1 \rightarrow y = \pm 1$

if $y = 1$ $x = 1$; if $y = -1$ $x = -1$

if $x = -y \Rightarrow y(-y) = 1 \Rightarrow y^2 = -1$ no real solutions

$\therefore$ the critical points are $(1,1), (-1,-1)$

$f(1,1) = 1^2 + 1^2 = 2$
$f(-1,-1) = (-1)^2 + (-1)^2 = 2$

all are the same $\therefore$ all are the same kind of extrema, either max or min

By inspection of the graph they end up being minimums.

8. a $\int_0^{\pi}\int_1^2 y\sin(xy)\,dx\,dy = \int_0^{\pi}\left(-y\frac{\cos(xy)}{y}\Big|_1^2\right)dy$

$= \int_0^{\pi}(-\cos(2y)+\cos y)\,dy = -\frac{1}{2}\sin 2y + \sin y\Big|_0^{\pi}$

$= -\frac{1}{2}\sin(2\pi) + \sin\pi + \frac{1}{2}\sin(0) - \sin(0) = 0$

b. $\int_0^2\int_{x^2}^{2x}(x^2+y^2)\,dy\,dx \leftarrow$ dy must be 1st

$= \int_0^2\left(x^2y + \frac{y^3}{3}\Big|_{x^2}^{2x}\right)dx = \int_0^2\left(2x^3 + \frac{8x^3}{3} - x^4 - \frac{x^6}{3}\right)dx$

$= \int_0^2\left(\frac{14}{3}x^3 - x^4 - \frac{x^6}{3}\right)dx = \frac{7x^4}{6} - \frac{x^5}{5} - \frac{x^7}{21}\Big|_0^2$

$= \frac{7(16)}{6} - \frac{32}{5} - \frac{128}{21} =$

$= \frac{7(8)(35) - 32(21) - 128(5)}{5(21)} = \frac{648}{105} = \frac{216}{35}$