Calc 2 Test #3 Review G. Bothusiem

1. Determine whether the series converge or diverge & why.

a. $\sum_{n=0}^{\infty} (1.67)^n$ b. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n}\right)$ c. $\sum_{n=1}^{\infty} \frac{6}{5n-1}$

d. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n}}$ e. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ f. $\sum_{n=1}^{\infty} \left(\frac{3n-1}{2n+5}\right)^n$

g. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ h. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

2. Determine whether #1e converges conditionally or absolutely.

3. Find the interval of convergence for the following power series.

a. $\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$ b. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(n+1)^2}$ c. $\sum_{n=0}^{\infty} n!(x-2)^n$

4. Find the power series centered at 0 for

$g(x) = \frac{2}{3-x}$

5. Find the power series for the derivative in #4.

6. Find the power series for $f(x) = \sin x$ centered at $c = 3\pi/4$

7. Find the power series for $g(x) = 3^x$ centered at $C = 0$

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1a. $\sum_{n=0}^{\infty} (1.67)^n$ Geometric Series $r = 1.67$ $|r| > 1$ $\therefore$ diverges

b. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1-n}{n^2}$ compare to $\sum_{n=1}^{\infty} \frac{-1}{n}$ Harmonic series Diverges

$\lim_{n\to\infty} \frac{1-n}{n^2} \cdot \frac{n}{-1} = 1$ Finite Positive

$\therefore$ Diverges by Limit Comparison Test

c. $\sum_{n=1}^{\infty} \frac{6}{5n-1}$ compare to $\sum \frac{1}{n}$ harmonic series Diverges

$\lim_{n\to\infty} \frac{6}{5n-1} \cdot \frac{n}{1} = \frac{6}{5}$ Finite Positive $\therefore$ Diverges as well Be Limit Comparison Test

d. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n}}$ compare to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ Converges p-series $p = 3/2 > 1$

$\frac{1}{\sqrt{n^3+2n}} \leq \frac{1}{\sqrt{n^3}} \rightarrow n^3+2n > n^3$ $\therefore$ converges as well be Direct Comparison Test

e. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$ $\lim_{n\to\infty} \frac{1}{n^5} = 0$ ; $\frac{1}{(n+1)^5} \leq \frac{1}{n^5}$

$\therefore$ converges by Alternating Series Test

f. $\sum_{n=1}^{\infty} \left(\frac{3n-1}{2n+5}\right)^n$ $\lim_{n\to\infty} \left|\sqrt[n]{\left(\frac{3n-1}{2n+5}\right)^n}\right| = \lim_{n\to\infty} \left|\frac{3n-1}{2n+5}\right| = 3/2$

$\frac{3}{2} > 1$ $\therefore$ Diverges by Root Test

1.g $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ use $f(x) = \frac{\ln x}{x}$ · pos. ✓ · cont. ✓ · Dec ✓

$$\int_1^{\infty} \frac{\ln x}{x} dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x} dx = \lim_{b\to\infty} \int_1^b u\,du = \lim_{b\to\infty} \frac{u^2}{2}\Big|_1^b$$

$u = \ln x$
$du = 1/x \, dx$

$= \lim_{b\to\infty} \frac{b^2}{2} - \frac{1}{2} = \infty$ Diverges ∴ the Series Diverges By Integral Test

h. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$ $\lim_{n\to\infty} \left| \frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right|$

$= \lim_{n\to\infty} \left| \frac{n+1}{e} \right| = \infty$ ∴ Diverges by Ratio Test

2. $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^5} \right| = \sum_{n=1}^{\infty} \frac{1}{n^5}$ p-series $p = 5 > 1$ ∴ Converges

∴ $\sum \frac{(-1)^n}{n^5}$ converges absolutely

3a. $\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$ $\lim_{n\to\infty} \left| \frac{x^{n+1}}{10^{n+1}} \cdot \frac{10^n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{10} \right| = \left| \frac{x}{10} \right|$

When $\left|\frac{x}{10}\right| < 1$ converges by Ratio Test ∴ the initial interval of convergence is $-10 < x < 10$

The
Check $x = 10$ $\sum_{n=0}^{\infty} \left(\frac{10}{10}\right)^n = \sum_{n=0}^{\infty} 1^n$ Geometric Series $r = 1$ ∴ Diverges

Check $x = -10$ $\sum_{n=0}^{\infty} \left(\frac{-10}{10}\right)^n = \sum_{n=0} (-1)^n$ Geometric $r = -1$ $|r| = 1$ ∴ Diverges

Interval of convergence $(-10, 10)$

3.b $\sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{(n+1)^2}$ $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-2)^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{(-1)^n(x-2)^n} \right|$

$= \lim_{n\to\infty} \left| x-2 \cdot \frac{(n+1)^2}{(n+2)^2} \right| = |x-2|$ when $<1$ converges by Ratio Test

$|x-2|<1 \to -1 < x-2 < 1 \to 1 < x < 3$

Check $x=1$ $\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{2n}}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2}$

Compare to $\sum \frac{1}{n^2}$ convergent p-series.

By limit Comparison Test $\sum \frac{1}{(n+1)^2}$ converges.

Check $x=3$ $\sum_{n=0}^{\infty} \frac{(-1)^n(1)^n}{(n+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$ $\lim_{n\to\infty} \frac{1}{(n+1)^2} = 0$

$\frac{1}{(n+2)^2} \le \frac{1}{(n+1)^2}$ $\therefore$ Converges by Alternating Series Test.

Interval of Convergence $[1, 3]$

3c. $\sum_{n=0}^{\infty} n!(x-2)^n$ $\lim_{n\to\infty} \left| \frac{(n+1)!(x-2)^{n+1}}{n!\,(x-2)^n} \right| = \lim_{n\to\infty} |(n+1)(x-2)|$

$= |x-2|\infty$ which is always $>1$ $\therefore$ Diverges for all $x$ except $x=2$

By Ratio test. $\therefore$ the series only converges at $x=2$.

4. $g(x) = \frac{2}{3-x} = \frac{2}{3(1-\frac{x}{3})} = \frac{2/3}{1 - x/3}$ let $a = 2/3$, $r = x/3$

then $g(x) = \sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} x^n$

5. $g'(x) = \frac{0(3-x) - 2(-1)}{(3-x)^2} = \frac{2}{(3-x)^2}$

$= \sum_{n=1}^{\infty} \frac{2n}{3^{n+1}} x^{n-1}$

6. $f(x) = \sin x \qquad f(\frac{3\pi}{4}) = \sqrt{2}/2$

$f'(x) = \cos x \qquad f'(3\pi/4) = -\sqrt{2}/2$

$f''(x) = -\sin x \qquad f''(3\pi/4) = -\sqrt{2}/2$

$f'''(x) = -\cos x \qquad f'''(3\pi/4) = \sqrt{2}/2$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(3\pi/4) = \sqrt{2}/2$

$\sin x = f(3\pi/4) + f'(3\pi/4)(x - 3\pi/4) + \frac{f''(3\pi/4)}{2!}(x - 3\pi/4)^2 + \frac{f'''(3/4)}{3!}(x - 3/4)^3$

$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x - \frac{3\pi}{4}) - \frac{\sqrt{2}}{2 \cdot 2!}(x - 3\pi/4)^2 + \frac{\sqrt{2}}{2 \cdot 3!}(x - 3\pi/4)^3 + \frac{\sqrt{2}}{2 \cdot 4!}(x - 3\pi/4)^4 + \cdots$

$= \sum_{n=0}^{\infty} \frac{\sin(\frac{3\pi}{4} + \frac{n\pi}{2})}{n!}(x - 3\pi/4)^n$

7. $g(x) = 3^x \qquad g(0) = 1$

$g'(x) = 3^x \ln 3 \qquad g'(0) = \ln 3$

$g''(x) = 3^x (\ln 3)^2 \qquad g''(0) = (\ln 3)^2$

$g'''(x) = 3^x (\ln 3)^3 \qquad g'''(0) = (\ln 3)^3$

$3^x = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$

$= 1 + \ln 3\, x + \frac{(\ln 3)^2}{2!}x^2 + \frac{(\ln 3)^3}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{(\ln 3)^n}{n!} x^n$