Calc 2 Test #2 Problem Set G. Buthusiem

1. Integrate

a. $\int x\sqrt{x^2-36}\,dx$ b. $\int x^2 e^{3x}\,dx$ c. $\int e^{2x}\sin 3x\,dx$

d. $\int \cos^3(\pi x-1)\,dx$ e. $\int \sin^4 x\,dx$ f. $\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx$

g. $\int \frac{x^3}{\sqrt{4+x^2}}\,dx$ h. $\int \frac{x-39}{x^2-x-12}$ i. $\int \frac{x^2+2x}{x^3-x^2+x-1}\,dx$

2. Evaluate the limit

a. $\lim\limits_{x\to 1} \frac{(\ln x)^2}{x-1}$ b. $\lim\limits_{x\to\infty} \frac{e^{2x}}{x^2}$ c. $\lim\limits_{x\to\infty} (\ln x)^{2/x}$

3. Evaluate the integrals if possible, if not state that the integral diverges.

a. $\int_0^{16} \frac{1}{\sqrt[4]{x}}\,dx$ b. $\int_1^{\infty} x^2 \ln x\,dx$ c. $\int_1^{\infty} \frac{\ln x}{x^2}\,dx$

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1. a. $\int x\sqrt{x^2-36}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2}+C = \frac{1}{3}(x^2-36)^{3/2}+C$

$u = x^2-36 \quad du = 2x\,dx$

b. $\int x^2e^{3x}dx \longrightarrow = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$

$u = x^2 \quad + \quad dv = e^{3x}$

$2x \quad - \quad \frac{1}{3}e^{3x}$

$2 \quad + \quad \frac{1}{9}e^{3x}$

$0 \quad \frac{1}{27}e^{3x}$

c. $\int e^{2x}\sin 3x\,dx \longrightarrow = \frac{1}{2}(\sin 3x)e^{2x} - \int \frac{3}{2}e^{2x}\cos 3x\,dx$

$u = \sin 3x \quad dv = e^{2x}dx$

$du = 3\cos 3x\,dx \quad v = \frac{1}{2}e^{2x}$

$u = \frac{3}{2}\cos 3x \quad dv = e^{2x}dx$

$du = -\frac{9}{2}\sin 3x\,dx \quad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \int \frac{9}{4}e^{2x}\sin 3x\,dx$

$= \frac{4}{13}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x\right) + C$

d. $\int \cos^3(\pi x-1)dx = \int \cos^2(\pi x-1)\cos(\pi x-1)dx$

$= \int \left(1-\sin^2(\pi x-1)\right)\cos(\pi x-1)dx$

$u = \sin(\pi x-1) \quad du = \pi\cos(\pi x-1)dx$

$= \frac{1}{\pi}\int(1-u^2)\,du = \frac{1}{\pi}\left(u - \frac{u^3}{3}\right) + C$

$= \frac{1}{\pi}\left(\sin(\pi x-1) - \frac{\sin^3(\pi x-1)}{3}\right) + C$

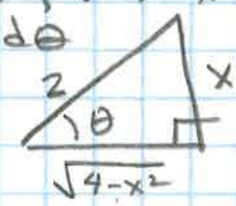
e. $\int \sin^4 x\,dx = \int \sin^2 x \sin^2 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right)dx$

$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \left(\frac{1+\cos 4x}{2}\right)\right)d\theta$

$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}\right)dx = \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right) + C$

f. $\int \frac{-12}{x^2\sqrt{4-x^2}}\,dx \longrightarrow = \int \frac{-12}{4\sin^2\theta\, 2\cos\theta}\, 2\cos\theta\, d\theta$

let $u = x$ $a = 2$ then if
$x = 2\sin\theta$, $\sqrt{4-x^2} = 2\cos\theta$
$dx = 2\cos\theta\, d\theta$



$= -3\int \frac{1}{\sin^2\theta}\,d\theta = -3\int \csc^2\theta\, d\theta$

$= +3\cot\theta + C$

$= 3\,\frac{\sqrt{4-x^2}}{x} + C$

g. $\int \frac{x^3}{\sqrt{4+x^2}}\,dx \longrightarrow = \int \frac{8\tan^3\theta}{2\sec\theta}\, 2\sec^2\theta\, d\theta$

let $u = x$ $a = 2$ then
$x = 2\tan\theta$ $\sqrt{4+x^2} = 2\sec\theta$
$dx = 2\sec^2\theta\, d\theta$

$= \int 8\tan^3\theta \sec\theta\, d\theta$

$= \int 8(\tan^2\theta)\sec\theta\tan\theta\, d\theta = \int 8(\sec^2\theta - 1)\sec\theta\tan\theta\, d\theta$

$u = \sec\theta$ $du = \sec\theta\tan\theta\, d\theta$

$= \int 8(u^2 - 1)\,du = 8\left(u^3/3 - u\right) + C$

$= 8\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C = 8\left(\frac{(4+x^2)^{3/2}}{24} - \frac{(4+x^2)^{1/2}}{2}\right) + C$

h. $\int \frac{x-39}{x^2-x-12}dx \longrightarrow \int\left(\frac{-5}{x-4}+\frac{6}{x+3}\right)dx$

$\frac{x-39}{(x-4)(x+3)} = \frac{A_1}{x-4} + \frac{A_2}{x+3}$

$= -5\ln|x-4| + 6\ln|x+3|$

$= \ln\left|\frac{(x+3)^6}{(x-4)^5}\right| + C$

$x-39 = A_1(x+3) + A_2(x-4)$

let $x=-3$ let $x=4$

$-42 = A_2(-7)$ $-35 = A_1(7)$

$A_2 = 6$ $A_1 = -5$

i. $\int \frac{x^2+2x}{x^3-x^2+x-1}dx \longrightarrow \int\left(\frac{3/2}{x-1}+\frac{-\frac{1}{2}x+\frac{3}{2}}{x^2+1}\right)dx$

$\frac{x^2+2x}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$

$\downarrow$ see below

$x^2+2x = A(x^2+1)+(Bx+C)(x-1)$

let $x=1$ $= Ax^2+A+Bx^2-Bx+Cx-C$

$3 = A(2)$ $= (A+B)x^2+(C-B)x+A-C$

$A = 3/2$ $A+B=1$ $C-B=2$ $A-C=0$

$B=-1/2$ $C=3/2$

$= \int\left(\frac{3/2}{x-1} - \frac{\frac{1}{2}x}{x^2+1} + \frac{3/2}{x^2+1}\right)dx$

$= 3/2\ln|x-1| - \frac{1}{4}\ln|x^2+1| + 3/2\arctan x + C$

$= \ln\left|\frac{(x-1)^{3/2}}{(x^2+1)^{1/4}}\right| + 3/2\arctan x + C$

2a. $\lim_{x\to 1} \frac{(\ln x)^2}{x-1} \to \frac{0}{0}$ use L'Hopital's Rule

$= \lim_{x\to 1} \frac{2(\ln x)\cdot\frac{1}{x}}{1} = \lim_{x\to 1} \frac{2\ln x}{x} = \frac{2(0)}{1} = 0$

b. $\lim_{x\to\infty} \frac{e^{2x}}{x^2} \to \frac{\infty}{\infty}$ use L'Hopital's Rule

$= \lim_{x\to\infty} \frac{2e^{2x}}{2x} \to \frac{\infty}{\infty}$ use L'Hop. Rule again

$= \lim_{x\to\infty} \frac{4e^{2x}}{2} = \infty$

c. $\lim_{x\to\infty} (\ln x)^{2/x} \to \infty^0$ use L'Hop. Rule

let $y=(\ln x)^{2/x} \Rightarrow \ln y = \frac{2}{x}\ln(\ln x)$

$= \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{2\ln(\ln x)}{x} = \lim_{x\to\infty} \frac{\frac{2}{x\ln x}}{1} = 0$

$\therefore \lim_{x\to\infty} y = e^0 = 1$

3. $\int_0^{16} \frac{1}{\sqrt[4]{x}}\,dx$ improper because discont at $x=0$

$= \lim_{b\to 0^+} \int_b^{16} \frac{1}{\sqrt[4]{x}}\,dx = \lim_{b\to 0^+} \int_b^{16} x^{-1/4}\,dx = \lim_{b\to 0^+} \left(\frac{4}{3} x^{3/4} \Big|_b^{16} \right)$

$= \lim_{b\to 0^+} \left(\frac{4}{3}(16)^{3/4} - \frac{4}{3}(b)^{3/4} \right) = \frac{4}{3}(8) - 0 = \frac{32}{3}$

b. $\int_1^{\infty} x^2 \ln x \, dx$ improper because of ∞ as upper limit

$= \lim_{b\to\infty} \int_1^b x^2 \ln x \, dx \longrightarrow = \lim_{b\to\infty} \left(\frac{x^3 \ln x}{3} - \int x^2/3 \, dx \right) \Big|_1^b$

$u = \ln x \quad dv = x^2 dx$
$du = 1/x \, dx \quad v = x^3/3$

$= \lim_{b\to\infty} \left(\frac{x^3 \ln x}{3} - \frac{x^3}{9} \right) \Big|_1^b = \lim_{b\to\infty} \left(b\left(\frac{\ln b}{3} - \frac{1}{9}\right) + \frac{1}{9} \right)$

$= \infty + 1/9 = \infty$ ∴ divergent

c. $\int_1^{\infty} \frac{\ln x}{x^2} dx \longrightarrow \lim_{b\to\infty} \left(-\frac{\ln x}{x} + \int \frac{1}{x^2} dx \right) \Big|_1^b$

let $u = \ln x \quad dv = x^{-2} dx$
$du = 1/x \, dx \quad v = -\frac{1}{x} dx$

$= \lim_{b\to\infty} \left(\frac{-\ln x}{x} - \frac{1}{x} \right) \Big|_1^b = \lim_{b\to\infty} \left(\frac{-\ln b - 1}{b} - \frac{0-1}{1} \right)$

$= \lim_{b\to\infty} \frac{-\ln b - 1}{b} + 1$

$= \lim_{b\to\infty} \frac{-\frac{1}{b}}{1} + 1 = 0 + 1 = +1$