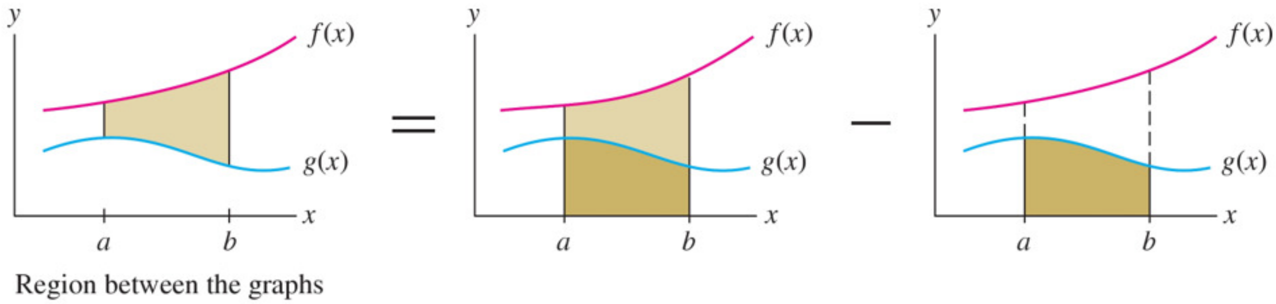
Applications of Integration
Area of a Region Between Two Curves

Objective: Find the area of a region between two curves using integration. Find the area of a region between intersecting curves using integration. Describe integration as an accumulation process.

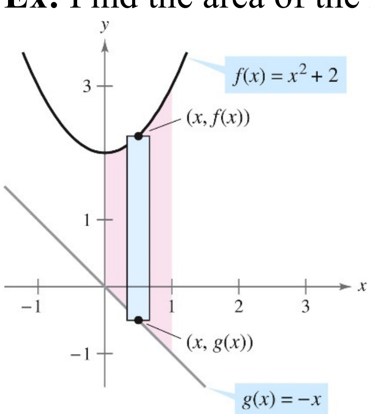
With a few modifications, we can extend the application of definite integrals from the area of a region under a curve to the area of a region between two curves.



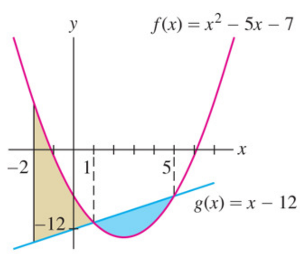
Area of a Region Between Two Curves:
If f and g are continuous on $[a,b]$ and $g(x) \leq f(x)$ for all x in [a,b], then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]dx$$

Ex: Find the area of the region bounded by the graphs of $y = x^2 + 2$, y= -x, x=0 and x=1.



Area of Region Between Intersecting Curves
When the graphs intersect the values for a and b must be computed and are the intersection points.
Ex: Find the area of the region bounded by the graphs of $f(x) = x^2 - 5x - 7$ and $g(x) = x - 12$



Ex: The sine and cosine curves intersect infinitely many times, bounding regions of equal areas. Find the area of one of these regions.

Curves that intersect at more than two points

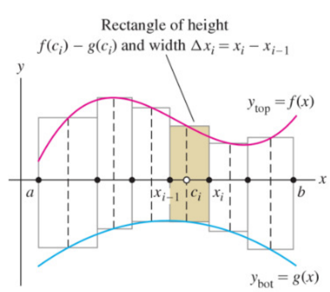
If two curves intersect at more than two points, then to find the area of the region between the curves you must find all points of intersection and check to see which curve is above the other in each interval determined by these points.

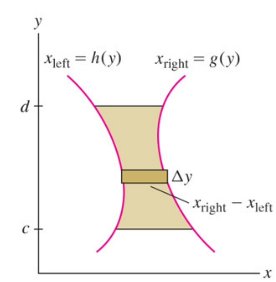
Ex: Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$

If the graph of a function of *y* is a boundary of a region, it is convenient to use representative rectangles that are horizontal and find the area by integrating with respect to *y*. In general, to determine are between two curves you can use

$$A = \int_{x=a}^{x=b} [(top\,curve) - (bottom\,curve)]dx \qquad A = \int_{y=c}^{y=d} [(right\,curve) - (left\,curve)]dy$$

(vertical rectangles) (horizontal rectangles)





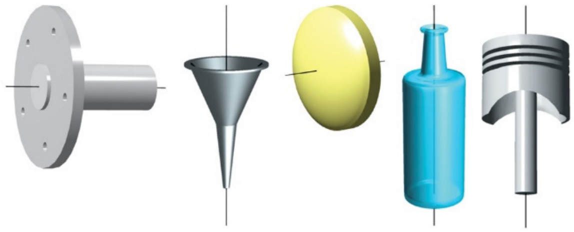
(B) Region between $x = h(y)$ and $x = g(y)$

Ex: Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x + y = 1$.

Volume: The Disk Method and Washer Method

Objective: Find the volume of a solid of revolution using the disk method. Find the volume of a solid of revolution using the washer method. Find the volume of a solid with known cross sections.

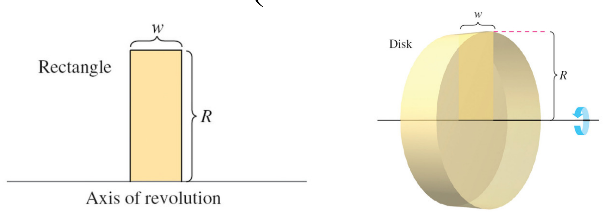
If a region in a plane is revolved about a line, the resulting solid is a **solid of revolution**, and the line is called the **axis of revolution**.



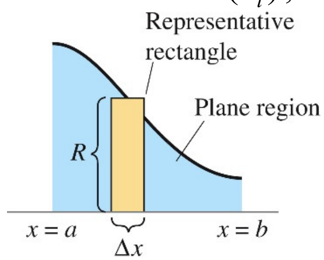
The simplest solid is a disk, a rectangle revolved about an axis, and the

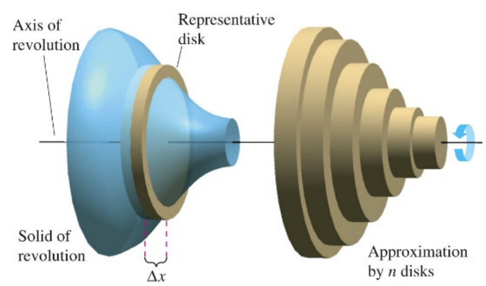
Volume of disk = (area of the circle)(width of the disk)

$= \pi R^2 w$ (where R is the radius of the disk)



To relate this to other solids we can approximate the solid using n such disks of width $\Delta x = \dfrac{b-a}{n}$ and with radius $R(x_i)$,





$$\textbf{Volume of solid} \approx \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x$$

This approximation becomes better if we let the number of disks n go to infinity

$$\textbf{Volume of solid} = \lim_{n\to\infty} \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x = \pi \int_a^b [R(x)]^2\, dx$$

The Disk Method:

To find the volume of a solid of revolution with the disk method, use one of the following:

Horizontal Axis of Revolution	Vertical Axis of Revolution
Volume = $V = \pi \int_a^b [R(x)]^2\, dx$	Volume = $V = \pi \int_c^d [R(y)]^2\, dy$

Ex: Find the volume of $f(x)=\sqrt{\sin x}$ the solid formed by revolving the region bounded by the graph of the x-axis. $(0 \le x \le \pi)$ about the x-axis.

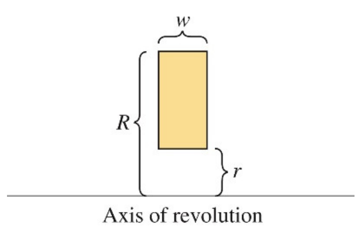
Ex: Find the volume of the solid formed by revolving the region bounded by $f(x)=2-x^2$ and g(x) = 1 about the line y = 1**.**

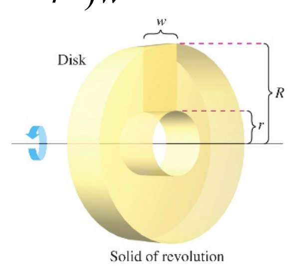
The Washer Method:

The disk method can be extended to cover solids of revolution with holes by replacing the representative disk with a representative washer.

Volume of a washer =(volume of the larger disk) – (volume of the hole)

$$=\pi R^2 w - \pi r^2 w = \pi(R^2 - r^2)w$$





Through the same methods as before by adding up n washers then letting n go to infinity, we get the **Washer Method:**

Horizontal Axis of Revolution

$$V=\pi\int_a^b ([R(x)]^2-[r(x)]^2)dx$$

where $[a,b]$ is an interval over the x-axis

Vertical Axis of Revolution

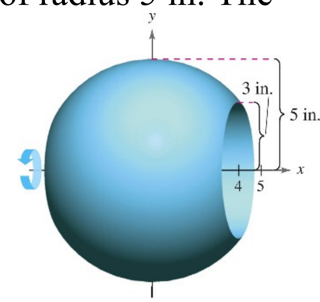
$$V=\pi\int_c^d ([R(y)]^2-[r(y)]^2)dy$$

where $[c,d]$ is an interval over the y-axis

Ex: Find the volume of the solid formed by revolving the region bounded by the graphs of $y=\sqrt{x}$ and $y=x^2$ about the x-axis.

Ex: Find the volume of the solid formed by revolving the region bounded by the graphs $y=x^2+1$, y =0, x =0, and x =1 about the y-axis.

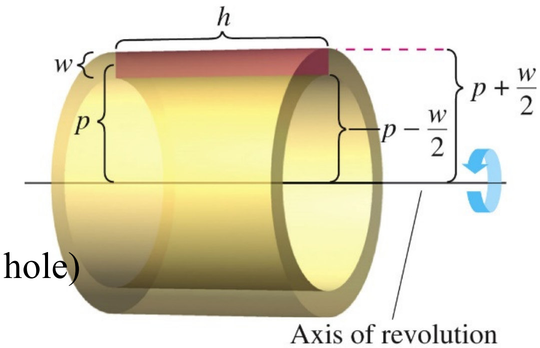
Ex: A manufacturer drills a hole through the center of a metal sphere of radius 5 in. The hole has radius 3 in. What is the volume of the resulting metal ring?



Solid of revolution

Volume: The Shell Method

Objective: Find the volume of a solid of revolution using the shell method. Compare the uses of the disk method and the shell method.

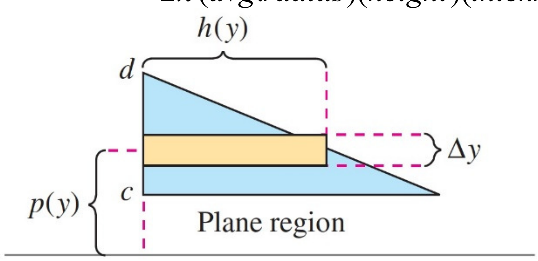


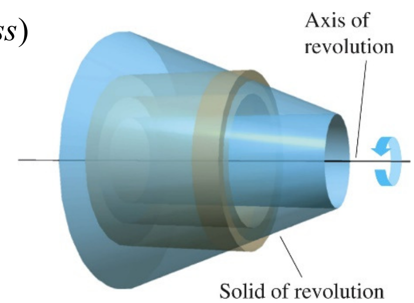
The advantage of the Shell Method is we can take rectangles **parallel** to the axis of revolution rather than perpendicular to it. First we need to look at the volume of a shell.

Volume of a Shell = (volume of cylinder) - (volume of hole)

$$=\pi\left(p+\frac{w}{2}\right)^2 h-\pi\left(p-\frac{w}{2}\right)^2 h=2\pi\, phw$$

$$=2\pi(avg.radius)(height)(thickness)$$





The Shell Method:

The volume of a solid obtained by rotating the region under the graph of $y = f(x)$ over the interval $[a,b]$ about the y-axis is equal to

$$V=2\pi\int_a^b (average\ radius)(height\ of\ shell)dx$$

If we rotate the graph of $x = f(y)$ over the interval $[c,d]$ about the x-axis the volume is equal to

$$V=2\pi\int_c^d (average\ radius)(height\ of\ shell)dy$$

Where the average radius is the distance from the axis of revolution to the center of the representative rectangles and the height of the shell is the height of the representative rectangle.

Differences between Disk/Washer Methods and Shell Method

- The **disk/washer methods** the representative **rectangles** are always **perpendicular** to the axis of revolution
- The **shell method** the representative **rectangles** are always **parallel** to the axis of revolution

Ex: Find the volume of the solid of revolution formed by revolving the region bounded by $y = x - x^3$ and the x-axis where $0 \le x \le 1$ about the y-axis.

Ex: Find the volume of the solid of revolution formed by revolving the region bounded by the graph of $x = e^{-y^2}$ and the y-axis where $0 \le y \le 1$ about the x-axis.

Ex: Find the volume of the solid of revolution formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

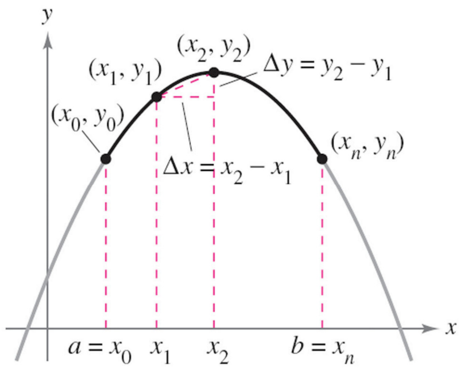
Ex: Find the volume of the solid of revolution formed by revolving the region bounded by the graph of $y = x^3 + x + 1, y = 1$, and $x = 1$ about the line $x = 2$

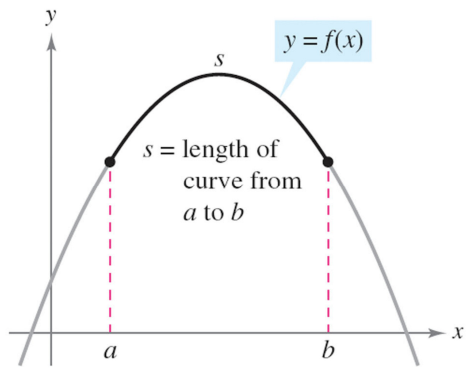
Arc Length and Surfaces of Revolution

To find the distance between two points recall the formula

$$d = \sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

This is a direct distance between points but what if we want to find the distance along a curve. We call this arc length.





Definition of Arc Length:

Let the function given by $y = f(x)$ represent a smooth curve on the interval $[a,b]$. The arc length of f between a and b is

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Similarly, for a smooth curve given by $x = g(y)$, the arc length of g between c and d is

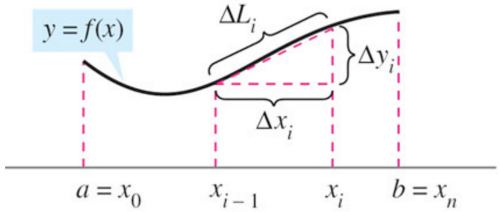
$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

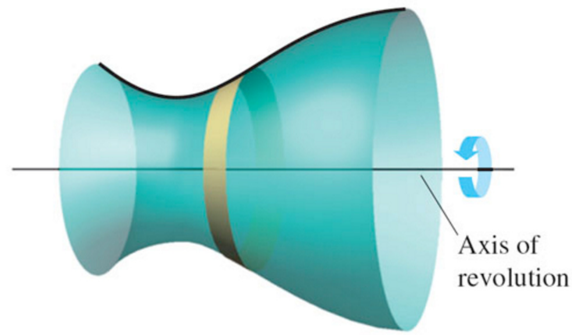
Ex: Find the arc length of $y = 2x^{3/2} + 3$ over $[0,8]$.

Ex: Find the arc length of the parabola $y^2 = x$ over from (0,0) to (1,1).

Surface of Revolution

If a graph of a continuous function is revolved around a line, the resulting surface is a **surface of revolution**.





Definition of the Area of a Surface of Revolution

Let $y = f(x)$ have a continuous derivative on the interval $[a,b]$. The area of the surface of revolution formed by revolving the graph of f about a the x-axis is

$$S = 2\pi\int_a^b r(x)\sqrt{1+[f'(x)]^2}\,dx$$

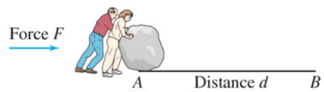
where r(x) is the distance between the graph of f and the axis of revolution. Same can be done with respect to y about a vertical axis.

Ex: Find the area of the surface formed by revolving the graph of $f(x) = x^3$ in the interval [0,1] about the x-axis.

Ex: The curve $y = \sqrt{4 - x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis.

Work

All physical task from running up a hill to turning on the computer, require an expenditure of energy. When force is applied to an object to move it, the energy expended is called **work**. When a constant force F is applied to move the object a distance d in the direction of the force, the work W is defined as "force times distance"



The work expended to move the object from A to B is $W = F \cdot d$.

$$W = F \bullet d$$

The SI unit of force is the *newton* (N), defined as 1 kg-m/s^2. Energy and work are both measured in units of the *joule* (J), equal to 1N-m. In the British system, the unit force is the *pound*, and both energy and work are measured in *foot-pounds* (ft-lb). Another unit of energy is the *calorie*. One ft-lb is approximately 0.738 J or 3.088 calories.

If a variable force is applied to an object, calculus is needed to determine the work.

Work done by Variable Force:
If an object is moved along a straight line by a continuously varying force $F(x)$, then the work W done by the force as the object is moved from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)dx$$

There are multiple methods of calculating force. Some of the most common ones are Hooke's Law (springs), Newton's Law of Universal Gravitation (attraction between particles), and Coulomb's Law (force between charges).

Hooks Law refers to the force F required to compress a spring and certain distance d.
$F = kd$ where k is the spring constant which depends on the nature of the spring.

Ex: Assuming a spring constant of $k = 400$ N/m, find the work required to

a. Stretch the spring 10 cm beyond equilibrium.
b. Compress the spring 2 cm more when it is already compressed 3 cm.