Applications of Derivatives

Extrema on an Interval

Objective: Understand the definition of extrema of a function on an interval. Understand the definition of relative extrema of a function on an open interval. Find extrema on a closed interval.

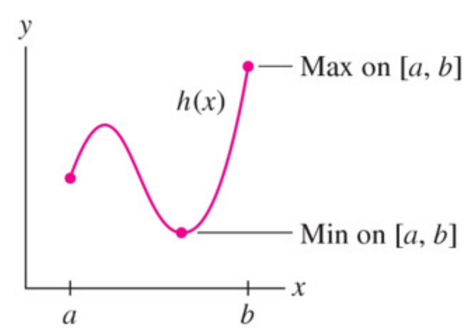
In calculus, much effort is devoted to determining the behavior of a function *f* on an interval *I*.
Does *f* have a maximum value on *I*? Does it have a minimum value? Where is the function increasing? Where is it decreasing?
In this chapter you will learn how derivatives can be used to answer these questions.

Definition of Extrema:
Let *f* be defined on an interval *I* containing *c*

1. *f(c)* is the **minimum of f** on *I* if $f(c) \leq f(x)$ for all *x* in *I*.
2. *f(c)* is the **maximum of f** on *I* if $f(c) \geq f(x)$ for all *x* in *I*.

The minimum and maximum of a function on an interval are the **extreme values**, or **extrema** of a function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum** on the interval.

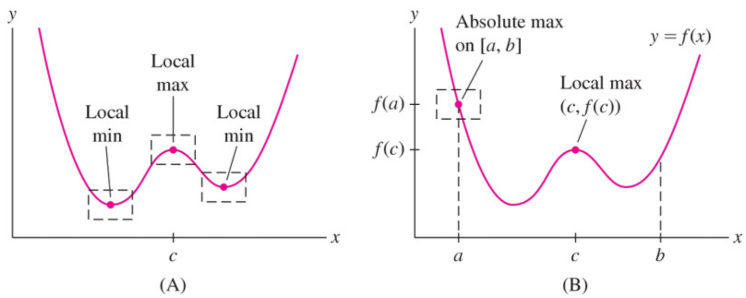
The Extreme Value Theorem:
If *f* is continuous on a closed interval [a,b], then *f* has both a minimum and a maximum in the interval.



Definition of Relative (Local) Extrema:

1. If there is an open interval containing c on which f(c) is a maximum, then f(c) is called a **relative (local) maximum** of f, or you can say that f has a **relative maximum at (c,f(c)).**
2. If there is an open interval containing c on which f(c) is a minimum, then f(c) is called a **relative (local) minimum** of f, or you can say that f has a **relative minimum at (c,f(c)).**

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minimum.

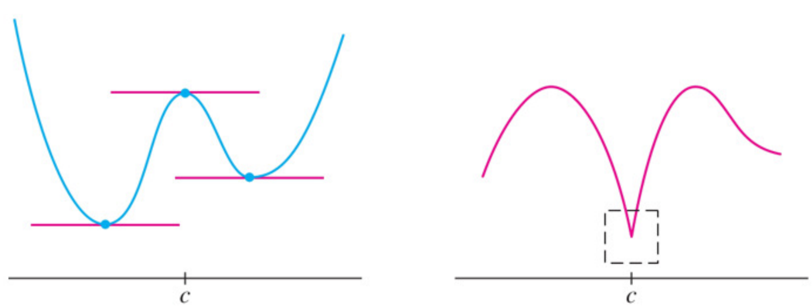


Ex: $\boldsymbol{f(x) = x^3 - 3x^2}$ has relative extrema at points (0,0) and (2,-4). Find the derivative at these points.

Ex: $\boldsymbol{f(x) = \sin x}$ has relative extrema at points (π/2,1) and (3π/2,-1) Find the derivative at these points.

<u>**Definition of a Critical Number:**</u>
Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a **critical number** of f.

<u>**Relative Extrema Occur Only at Critical Numbers (Fermat's Theorem):**</u>
If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.



(A) Tangent line is horizontal at the local extrema.

(B) This local minimum occurs at a point where the function is not differentiable.

How to Find Absolute Extrema on a Closed Interval:

Let f be a continuous function on a closed interval [a,b],

1. Find the critical numbers of f in (a,b).

2. Evaluate f at each critical number in (a,b).

3. Evaluate f at each endpoint of [a,b].

4. The least of these values in the minimum and the greatest is the maximum.

Ex: Find the absolute extrema of $f(x) = x^3 - 3x^2 + 1$ on the interval $-\frac{1}{2} \leq x \leq 4$.

Ex: Find the absolute extrema of $f(t) = 2\cos t + \sin 2t$ on the interval $\left[0, \frac{\pi}{2}\right]$.

Rolle's Theorem and the Mean Value Theorem

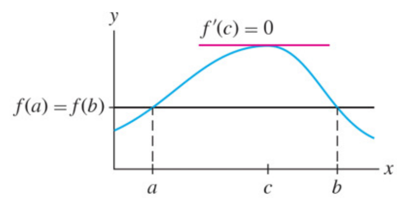
Objective: Understand and use Rolle's Theorem. Understand and use the Mean Value Theorem

Rolle's Theorem:

Let f be continuous on a closed interval [a,b] and differentiable on an open interval (a,b). If

f(a) = f(b)

then there is at least one number c in (a,b) such that $f'(c) = 0$.



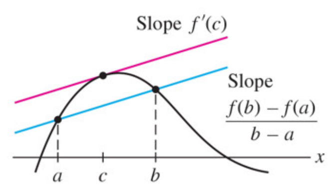
Ex: Find the two x intercepts of $f(x) = x^2 - 3x + 2$ and show that $f'(x) = 0$ at some point between that two intercepts.

Ex. Let $f(x) = x^4 - 2x^2$ Find all values of c in the interval (-2,2) such that $f'(c) = 0$ if they exist based on Rolle's Theorem.

The Mean Value Theorem:

If f is continuous on a closed interval [a,b] and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



(In simpler terms the average slope over [a,b] will be equal to the instantaneous slope at some point between a and b. This also could be thought of in terms of velocity)

Ex. Given that $f(x) = 5 - (4/x)$, find all values of c in the open interval (1,4) such that $f'(c) = \frac{f(4) - f(1)}{4 - 1}$.

Ex: Two stationary patrol cars equipped with radar are 5 miles apart on a highway. As a truck passes the first patrol car, its speed is clocked at
55 mph. Four minutes later, when the truck passes the second patrol car, its speed is clocked at 50 mph. Prove that the truck must have exceeded the speed limit of 55 mph at some time during the 4 minutes.
(Hint: 4 min = 1/15 hr.)

Increasing and Decreasing Functions and the First Derivative Test

Objective: Determine intervals on which a function is increasing or decreasing. Apply the First Derivative Test to find relative extrema of a function

Definitions of Increasing and Decreasing Functions:

- A function f is **increasing** of an interval if for any two numbers x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2)$$

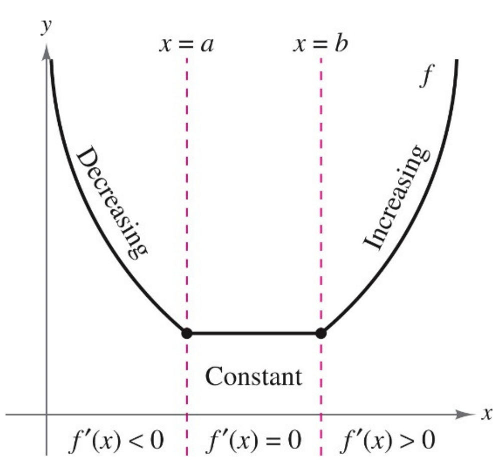
- A function f is **decreasing** of an interval if for any two numbers x_1 and x_2 in the interval,

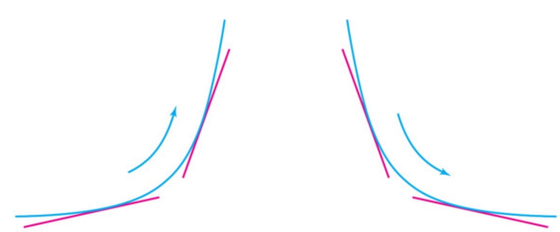
$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2)$$

Test for Increasing and Decreasing Functions:

Let f be a function that is continuous on the closed interval [a,b] and differentiable on the open interval (a,b).

- If $f'(x) > 0$ for all x in (a,b), then f is **increasing** on [a,b] .
- If $f'(x) < 0$ for all x in (a,b), then f is **decreasing** on [a,b] .
- If $f'(x) = 0$ for all x in (a,b), then f is **constant** on [a,b].





Increasing function: Tangent lines have positive slope.

Decreasing function: Tangent lines have negative slope.

How to Find Intervals on Which a Function Is Increasing of Decreasing:

Let f be continuous on the interval (a,b):

1. Locate the critical numbers of f in (a,b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use the sign of $f'(x)$ to determine whether f is increasing, decreasing, or constant on each interval.

These guidelines are also valid if the interval (a,b) is replaced by an interval of the form (-∞,b), (a,∞), or (-∞,∞).

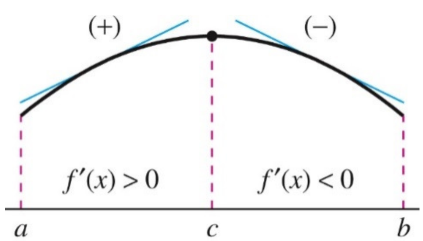
A function is ***strictly monotonic*** on an interval if it is either increasing on the entire interval or decreasing on the entire interval.

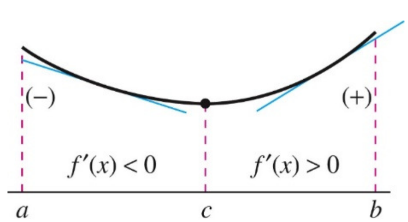
Ex: Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing and decreasing.

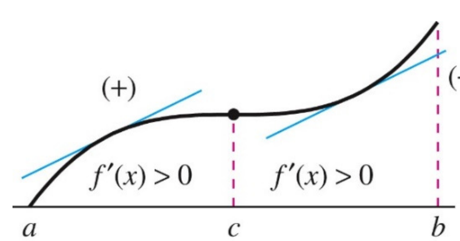
The First Derivative Test:

Let c be a **critical number** of a function f that is **continuous** on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then f(c) can be classified as follows.

- If $f'(x)$ changes from **negative** to **positive** at c, then f has a **relative minimum** at (c,f(c)).
- If $f'(x)$ changes from **positive** to **negative** at c, then f has a **relative maximum** at (c,f(c)).
- If $f'(x)$ is positive on both sides of c or negative on both sides of c, the f(c) is neither a relative minimum nor relative maximum.







Ex. Find the relative extrema of the function $f(x) = \frac{1}{2}x - sinx$ in the interval $(0,2\pi)$.

Ex. Find the relative extrema of $f(x) = (x^2 - 4)^{2/3}$

Ex. Find the relative extrema of $f(x) = \frac{x^4+1}{x^2}$

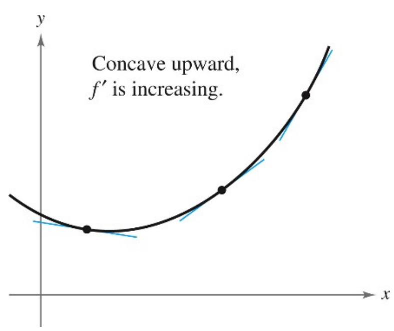
Concavity and the Second Derivative Test

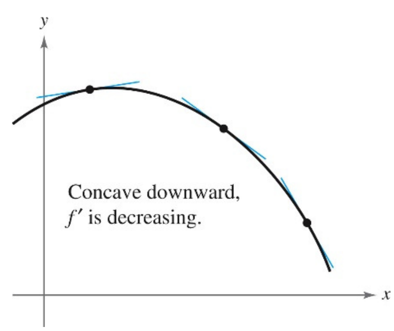
Objective: Determine intervals on which a function is concave upward or concave downward. Find any points of inflection of the graph of a function. Apply the Second Derivative Test to find relative extrema of a function

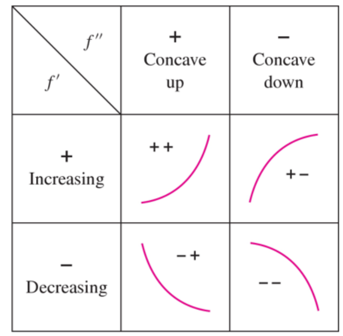
Definition of Concavity:

Let f be differentiable on an open interval I. The graph of f is

- **concave upward** on I if f' is increasing on the interval
- **concave downward** on I if f' is decreasing on the interval.





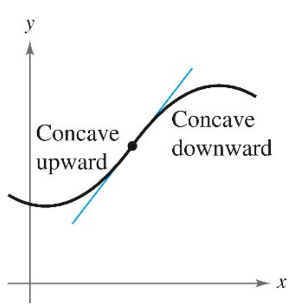


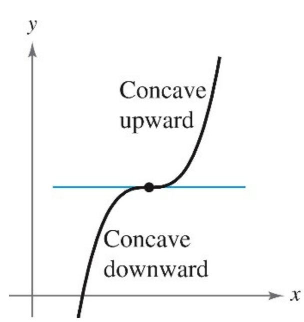
Graphical interpretation:

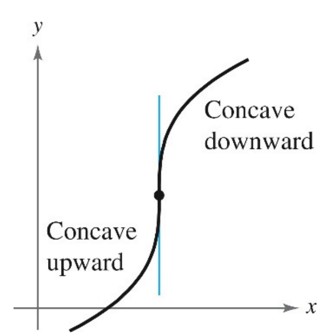
1. If the graph of f is **concave upward** on I, then the graph of f lies **above** all of its tangent lines.
2. If the graph of f is **concave downward** on I, then the graph of f lies **below** all of its tangent lines.

Definition of Point of Inflection:

Let f be a function that is continuous on an open interval and let c be a point in the interval. The point $(c, f(c))$ is a **point of inflection** of the graph of f if the concavity of changes at the point.







Points of Inflection:

If (c, f(c)) is a point of inflection of graph of f, then either $f''(c) = 0$ or $f''(c)$ does not exist at $x = c$.

<u>Test for Concavity:</u>
Let f be a function whose second derivative exists on an open interval *I*.
1. If $f''(x) > 0$ for all x in *I*, then the graph of f is **concave upward** in *I*.
2. If $f''(x) < 0$ for all x in *I*, then the graph of f is **concave downward** in *I*.

Ex: Determine the open intervals on which the graph of $f(x) = \dfrac{6}{(x^2+3)}$ is concave upward or downward.

Ex: Determine the open intervals on which the graph of $f(x) = \dfrac{(x^2+1)}{(x^2-4)}$ is concave upward or downward.

Ex: Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^2$.

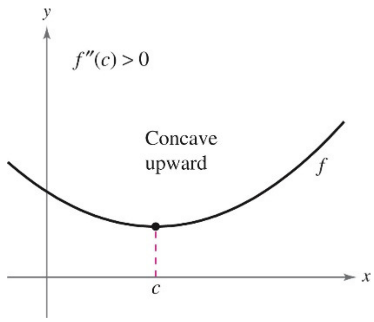
Second Derivative Test:

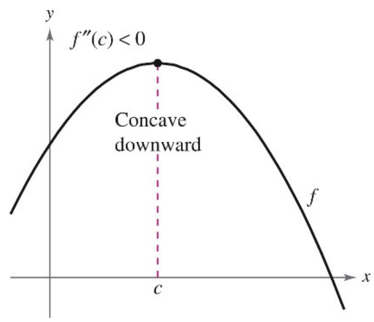
Let f be a function such that $f'(c)=0$ or does not exist and the second derivative of f exists on an open interval containing c.

1. If $f''(c)>0$, then f has a relative minimum at (c, f(c)).

2. If $f''(c)<0$, then f has a relative maximum at (c, f(c)).

If $f''(c)=0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.



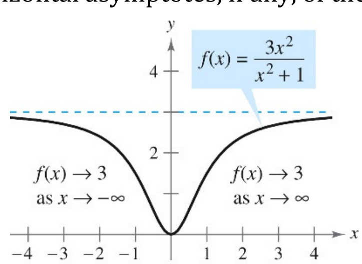


Ex: Use the second derivative test if possible to find the relative extrema for $f(x)=-3x^5+5x^3$.

Limits at Infinity

Objective: Determine (finite) limits at infinity. Determine the horizontal asymptotes, if any, of the graph of a function. Determine infinite limits at infinity.

Look at the end behavior of $f(x)=\dfrac{3x^2}{x^2+1}$



Graphically you will see that as x increases or decreases without bound f(x) approaches 3 (the function has a horizontal asymptote on both the left and right).

Also look at a table:

x	$-\infty\leftarrow$	-100	-10	-1	0	1	10	100	$\rightarrow\infty$
f(x)	3$\leftarrow$	2.9997	2.97	1.5	0	1.5	2.97	2.9997	$\rightarrow$3

So we write

$\lim\limits_{x\to\infty} f(x)=3$ and $\lim\limits_{x\to-\infty} f(x)=3$

Definition of a Horizontal Asymptote.

The line $y = L$ is a **horizontal asymptote** of the graph of $f\,(\mathrm{x})$ if as

$$x \to -\infty \text{ or } x \to \infty \text{ then } f(x) \to L$$

or

The line $y = L$ is a **horizontal asymptote** of the graph of f if

$$\lim_{x \to -\infty} f(x) = L \text{ or } \lim_{x \to \infty} f(x) = L$$

Limits at Infinity

If r is a positive rational number and c is any real number, then

$$\lim_{x \to \infty} \frac{c}{x^r} = 0 \text{ and } \lim_{x \to -\infty} \frac{c}{x^r} = 0$$

Ex. Find the limit: $\lim_{x \to \infty} \left(5 - \frac{2}{x^2}\right)$

Ex. Find each limit

a. $\lim_{x \to \infty} \frac{2x+5}{3x^2+1}$ **b.** $\lim_{x \to -\infty} \frac{2x^2+5}{3x^2+1}$ **c.** $\lim_{x \to -\infty} \frac{2x^3+5}{3x^2+1}$

How to Find Limits at ±∞ of Rational Functions:

1. If the **degree of the numerator is less than the degree of the denominator**, then the **limit** of the rational function as $x \to \pm\infty$ **is 0.**

2. If the **degree of the numerator is equal to the degree of the denominator**, then the **limit** of the rational function as $x \to \pm\infty$ **is** the **ratio of the leading coefficients.**

3. If the **degree of the numerator is greater than the degree of the denominator**, then the **limit** of the rational function as $x \to \pm\infty$ **does not exist**, but you may still be able to give an answer of ∞ or -∞.

Ex. Find each limit:

a. $\lim_{x \to \infty} \frac{3x-2}{\sqrt{2x^2+1}}$ **b.** $\lim_{x \to -\infty} \frac{3x-2}{\sqrt{2x^2+1}}$ **c.** $\lim_{x \to \infty} \sin x$

A Summary of Curve Sketching

Objective: Analyze and sketch the graph of a function. Concepts used in analyzing the graph of a function; x and y intercepts, symmetry, domain and range, continuity, vertical asymptotes, differentiability, relative extrema, concavity, points of inflection, horizontal asymptotes, infinite limits at infinity.

How to Analyze the Graph of a Function

1. Determine the domain and range of a function
2. Determine the intercepts, asymptotes, and symmetry of a graph
3. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine increasing, decreasing, relative extrema, concavity, and points of inflection.

Ex. Analyze and sketch the graph of $f(x) = \frac{2(x^2-9)}{x^2-4}$

Ex. Analyze and sketch the graph of $f(x) = \frac{x^2-2x+4}{x-2}$

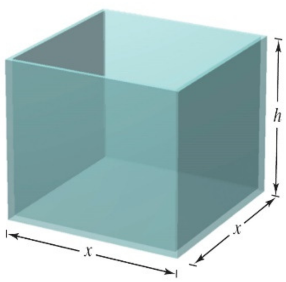
Ex. Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$

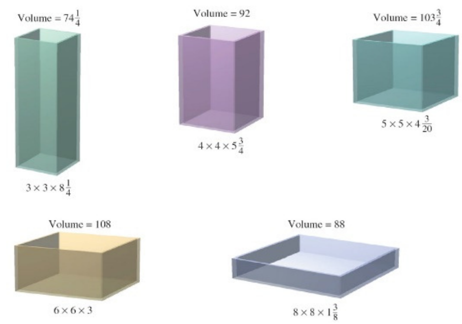
Optimization Problems

Objective: Solve applied minimum and maximum problems

One of the most common applications of calculus involves the determination of minimum and maximum values. Terms like: greatest profit, least cost, optimum size, and greatest distance are all terms that refer to max and min's.

Ex: A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



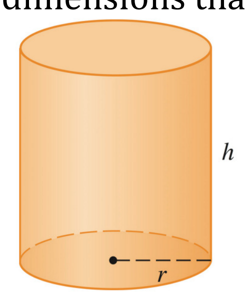


Primary Equation: gives the formula for the quantity to be optimized.
Secondary Equation: relates independent variables of the primary equation.

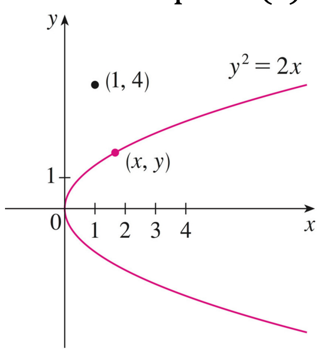
How to Solve Applied Minimum and Maximum Problems:

1. Identify all given quantities and quantities to be determines. If possible, make a sketch.
2. Write a primary equation for the quantity that is to be maximized or minimized.
3. Reduce the primary equation to one having a single independent variable. This may involve the use of a secondary equation.
4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5. Determine the desired maximum or minimum value by calculus techniques discussed previously in the chapter.

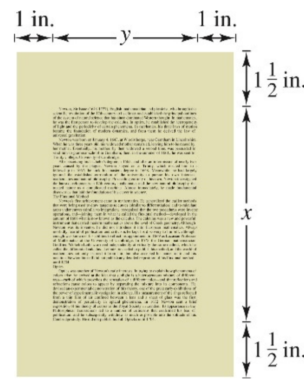
Ex: Campbell's soup wants to make a can that holds 12oz of soup. Find the dimensions that will minimize the cost of the metal to manufacture the can.



Ex: Which point(s) on the graph of $y^2 = 2x$ are closest to the point (1,4)?



Ex: A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1½ inches and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?



Ex: Find the area of the largest rectangle that can be inscribed in a semicircle of radius r.