Antiderivatives and The Integral

Antiderivatives

Objective: Use indefinite integral notation for antiderivatives. Use basic integration rules to find antiderivatives.

Another important question in calculus is given a derivative find the function that it came from. This is the process known as integration.

Definition of an Antiderivative:
A function \( F \) is an antiderivative of \( f \) on an interval \( I \) if \( F'(x) = f(x) \) for all \( x \) in \( I \).

Representation of Antiderivatives:
If \( F \) is an antiderivative of \( f \) on an interval \( I \), then \( G \) is an antiderivative of \( f \) on the interval \( I \) if and only if \( G \) is of the form \( G(x) = F(x) + C \), for all \( x \) in \( I \) where \( C \) is a constant.

\[ G(x) = F(x) + C \]

is called a family of antiderivatives or general antiderivative.

\( C \) is called the constant of integration
\( G \) is also know as the solution to the differential equation
A differential equation in \( x \) and \( y \) is an equation that involves \( x \), \( y \), and derivatives of \( y \).

Ex: Find the general solution of the differential equation \( y' = 2 \)

Notation for Antiderivatives
The process of finding antiderivatives is called antidifferentiation or indefinite integration and is denoted by an integral sign: \( \int \)

So from \( \frac{dy}{dx} = f(x) \) \( \Rightarrow \) \( dy = f(x)dx \)

using integration on both sides of the equation

\[ \int dy = \int f(x)dx = F(x) + C \]

this is the indefinite integral

Since integration is the reverse of differentiation we can check the previous by \( \frac{d}{dx}[F(x) + C] = f(x) \)

If you know your derivative rules then learning your integration rules should be very easy! Just work backwards.
**Basic Integration Rules:**

<table>
<thead>
<tr>
<th>Differentiation Formula</th>
<th>Integration Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{d}{dx}[C] = 0 )</td>
<td>( \int 0 , dx = C )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[kx] = k )</td>
<td>( \int k , dx = kx + C )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[kf'(x)] = kf''(x) )</td>
<td>( \int kf'(x) , dx = k \int f'(x) , dx )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x) )</td>
<td>( \int [f(x) \pm g(x)] , dx = \int f(x) , dx \pm \int g(x) , dx )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[x^n] = nx^{n-1} )</td>
<td>( \int x^n , dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1 )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\sin x] = \cos x )</td>
<td>( \int \cos x , dx = \sin x + C )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\cos x] = -\sin x )</td>
<td>( \int \sin x , dx = -\cos x + C )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\tan x] = \sec^2 x )</td>
<td>( \int \sec^2 x , dx = \tan x + C )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\sec x] = \sec x \tan x )</td>
<td>( \int \sec x \tan x , dx = \sec x + C )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\cot x] = -\csc^2 x )</td>
<td>( \int \csc^2 x , dx = -\cot x + C )</td>
</tr>
<tr>
<td>( \frac{d}{dx}[\csc x] = -\csc x \cot x )</td>
<td>( \int \csc x \cot x , dx = -\csc x + C )</td>
</tr>
</tbody>
</table>

**Ex:**

a. \( \int 3x \, dx \)  
 b. \( \int \frac{1}{x^3} \, dx \)  
 c. \( \int \sqrt{x} \, dx \)  
 d. \( \int 2 \sin x \, dx \)  
 e. \( \int \, dx \)  
 f. \( \int (x + 2) \, dx \)  
 g. \( \int 3x^4 - 5x^2 + x \, dx \)  
 h. \( \int \frac{x+1}{\sqrt{x}} \, dx \)  
 i. \( \int \frac{\sin x}{\cos^2 x} \, dx \)
Area:
Objective: Use sigma notation to write and evaluate a sum. Understand the concept of area. Approximate the area of a plane region. Find the area of a plane region using limits.

Sigma Notation:
The sum of \( n \) terms \( a_1, a_2, a_3, \ldots, a_n \) is written as
\[
\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n
\]
where \( i \) is the index of summation, \( a_i \) is the \( i \)th term of the sum, and the upper and lower bounds of summation are \( n \) and \( 1 \).

Ex: a. \( \sum_{i=1}^{6} i \)  
b. \( \sum_{k=1}^{n} \frac{1}{n} (k^2 + 1) \)  
c. \( \sum_{i=1}^{n} f(x_i) \Delta x \)

Properties of Summations:
1. \( \sum_{i=1}^{n} ka_i = k \sum_{i=1}^{n} a_i \)
2. \( \sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \)

Summation Formulas:
1. \( \sum_{i=1}^{n} c = cn \)
2. \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \)
3. \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \)
4. \( \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \)

Ex: Evaluate \( \sum_{i=1}^{n} \frac{i + 1}{n^2} \) for \( n = 10, 100, 1000, 10000 \)

Area of a Plane Region
Use five rectangles to find two approximations of the area of the region lying between the graph of \( f(x) = x^2 \) and the \( x \)-axis between \( x = 0 \) and \( x = 2 \).

Rectangles outside the curve are called circumscribed rectangles and the sum of the areas is called the upper sum.
Rectangles inside the curve are called **inscribed rectangles** and the sum of the areas is called the **lower sum**.

For any region under a curve \( f \) bounded by the **x-axis** between \( x = a \) and \( x = b \).

\[ 1 \] The left end of the rectangle touches the curve \( = \sum_{i=1}^{n} f(m_i) \Delta x \)

\[ 2 \] The right end of the rectangle touches the curve \( = \sum_{i=1}^{n} f(M_i) \Delta x \)

where

- \( \Delta x = \frac{b-a}{n} \), \( n \) is the number of subintervals
- \( f(m_i) = f(a + (i - 1) \Delta x) \)
- \( f(M_i) = f(a + i \Delta x) \)

if the function in increasing or decreasing will change whether (1) or (2) are upper or lower sums

\( f(m_i) \) is an upper sum if \( f \) is decreasing and a lower if \( f \) is increasing

\( f(M_i) \) is a lower sum if \( f \) is decreasing and an upper if \( f \) is increasing

<table>
<thead>
<tr>
<th>Limits of the Lower and Upper Sums:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition of the Area of a Region in the Plane:</strong></td>
</tr>
<tr>
<td>Let ( f ) be continuous and nonnegative on the interval ([a,b]). The limits as ( n \to \infty ) of both the lower and upper sums exists and are equal to each other.</td>
</tr>
</tbody>
</table>

| **Ex:** Find the area of the region bounded by the graph \( f(x) = x^3 \), the **x–axis**, and the vertical lines \( x = 0 \) and \( x = 1 \). |
Riemann Sums and Definite Integrals

Objective: Understand the definition of a Riemann sum. Evaluate a definite integral using limits. Evaluate a definite integral using properties of definite integrals.

**Definition of Riemann Sum:**
Let $f$ be defined on the closed interval $[a,b]$, and let $\Delta$ be a partition of $[a,b]$ given by

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

where $\Delta x_i$ is the width of the $i$th subinterval. If $c_i$ is any point in the $i$th subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i) \Delta x_i, \quad x_{i-1} < c_i < x_i$$

is called the **Riemann Sum** of $f$ for the partition $\Delta$.

**Definition of a Definite Integral:**
If $f$ is defined on the closed interval $[a,b]$ and the limit

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x_i$$

exists, then $f$ is integrable on $[a,b]$ and the limit is denoted by

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_{a}^{b} f(x) \, dx$$

The limit is called the definite integral of $f$ from $a$ to $b$. The number $a$ is the **lower limit** of integration and the number $b$ in the **upper limit** of integration.

Notice the similarities between the definite integral and the indefinite integral. Even though they are similar, there is a major difference: the definite integral results in a number, and the indefinite integral results in a family of functions.

**Ex:** Evaluate the definite integral $\int_{-2}^{1} 2x \, dx$ remember $x_i = \Delta x = \frac{b-a}{n}$ and $c_i = a + i(\Delta x)$

**Continuity Implies Integrability:**
If a function $f$ is continuous on the closed interval $[a,b]$, then $f$ is integrable on $[a,b]$. 
The Definite Integral as the Area of a Region:
If $f$ is continuous and nonnegative on the closed interval $[a,b]$, then the area of the region bounded by the graph of $f$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b f(x) \, dx$$

**Ex:** Sketch the region corresponding to the definite integral: $\int_1^4 dx$

Definitions of Two Special Integrals:
1. If $f$ if defined at $x = a$, then we define $\int_a^a f(x) \, dx = 0$
2. If $f$ is integrable on $[a,b]$, then we define $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$

Additive Interval Property:
If $f$ is integrable on the three closed intervals $[a,c],[c,b]$, and $[a,b]$ then,

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$$

Properties of Definite Integrals:
If $f$ and $g$ are integrable on $[a,b]$ and $k$ is constant, then the functions of $kf$ and $f \pm g$ are integrable on $[a,b]$, and

1. $\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$
2. $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

The Fundamental Theorem of Calculus
Objective: Evaluate a definite integral using the Fundamental Theorem of Calculus. Understand and use the Mean Value Theorem for Integrals. Find the average value of a function over a closed interval. Understand and use the Second Fundamental Theorem of Calculus.

We have looked at two major branches of calculus: differential calc (tangent line problem) and integral calc. (area problem). Even though the two seem unrelated there is a connection called the Fundamental Theorem of Calculus.

If a function $f$ is continuous on the closed interval $[a,b]$ and $F$ is an antiderivative of $f$ on the interval $[a,b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$
Using the Fundamental Theorem of Calculus

1. Provided you can find an antiderivative of $f$, you now have a way to evaluate a definite integral without having to use the limit of sum.

2. When applying the fundamental Theorem of Calculus, the following notation is convenient.

$$\int_{a}^{b} f(x)dx = F(x)\bigg|_{a}^{b} = F(b) - F(a)$$

3. It is not necessary to include a constant of integration $C$ in the antiderivative because

$$\int_{a}^{b} f(x)dx = [F(x) + C]_{a}^{b}$$

$$= [F(b) + C] - [F(a) + C]$$

$$= F(b) - F(a)$$

Ex: Evaluate each definite integral

a. $\int_{1}^{2} (x^2 - 3)dx$

b. $\int_{1}^{4} \sqrt{x}dx$

c. $\int_{0}^{\pi/4} \sec^2 x dx$

d. $\int_{0}^{2} |2x - 1| dx$

Ex: Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$.

The Second Fundamental Theorem of Calculus:

If $f$ is continuous on an open interval $I$ containing $a$, then, for every $x$ in the interval,

$$\frac{d}{dx} \left[ \int_{a}^{x} f(t)dt \right] = f(x)$$

Ex: Evaluate $\frac{d}{dx} \int_{0}^{x} \cos t dt$

Integration by Substitution

Objective: Use pattern recognition to find an indefinite integral. Use a change of variables to find an indefinite integral. Use the General Power Rule for Integration to find an indefinite integral. Use a change of variables to evaluate a definite integral. Evaluate a definite integral involving an even or odd function

Pattern Recognition:

We will look at integrating composition functions in two ways pattern recognition and change of variables.

Remember the Chain Rule: $\frac{d}{dx} [f(g(x))] = f'(g(x))g'(x)$
**Anti-differentiation of a Composite Function:**
Let $g$ be a function whose range is an interval $I$, and let $f$ be a function that is continuous on $I$, If $g$ is differentiable on its domain and $F$ is an antiderivative of $f$ on $I$, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

If $u = g(x)$ then $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C$$

Recognize the patterns that the following fit $f(g(x))g'(x)$

**Ex:**
- a. $\int 2x(x^2 + 1)^4 dx$
- b. $\int 3x^2\sqrt{x^3 + 1}dx$
- c. $\int \sec^2 x(tan x + 3)dx$
- d. $\int (x^2 + 1)^2 2xdx$
- e. $\int 5\cos 5xdx$
- f. $\int x(x^2 + 1)^2 dx$

**Making a Change of Variables**
1. Choose a substitution $u = g(x)$. Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)dx$
3. Rewrite the integral on terms of the variable $u$.
4. Find the resulting integral in terms of $u$.
5. Replace $u$ by $g(x)$ to obtain an antiderivative in terms of $x$.
6. Check your answer by differentiating.

**Ex:**
- a. $\int \sqrt{2x - 1}dx$
- b. $\int x\sqrt{2x - 1}dx$
- c. $\int \sin^2 3x \cos 3xdx$

**General Power Rule for Integration:**
If $g$ is a differentiable function of $x$, then

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

Equivalently, if $u = g(x)$, then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

**Ex:**
- a. $\int 3(3x - 1)^4 dx$
- b. $\int (2x + 1)(x^2 + x)dx$
- c. $\int 3x^2\sqrt{x^3 - 2}dx$
- d. $\int \frac{-4x}{(1-2x^2)^2}dx$
- e. $\int \cos^2 x \sin x dx$
Change of Variables for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a,b]$ and $f$ is continuous on the range of $g$, then

$$
\int_{a}^{b} f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du
$$

Ex:  

a. $\int_{0}^{1} x(x^2 + 1)^3\,dx$  

b. $\int_{1}^{5} \frac{x}{\sqrt{2x-1}}\,dx$