Antiderivatives and The Integral

Antiderivatives

Objective: Use indefinite integral notation for antiderivatives. Use basic integration rules to find antiderivatives.

Another important question in calculus is given a derivative find the function that it came from. This is the process known as integration.

Definition of an Antiderivative:
A function **F** is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Representation of Antiderivatives:
If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form **G(x) = F(x) + C**, for all x in I where C is a constant.

G(x) = F(x) + C is called a **family of antiderivatives** or **general antiderivative**.
C is called the **constant of integration**
G is also know as the *solution* to the *differential equation*
A **differential equation** in x and y is an equation that involves x, y, and derivatives of y.

Ex: Find the general solution of the differential equation $y' = 2$

Notation for Antiderivatives

The process of finding antiderivatives is called **antidifferentiation** or **indefinite integration** and is denoted by an integral sign: $\int$

So from $\frac{dy}{dx} = f(x) \quad \Rightarrow \quad dy = f(x)dx$

using integration on both sides of the equation

$$\int dy = \int f(x)dx = F(x) + C$$

this is the **indefinite integral**

Since integration is the reverse of differentiation we can check the previous by $\frac{d}{dx}\left[F(x)+C\right] = f(x)$

If you know your derivative rules then learning your integration rules should be very easy! Just work backwards.

Basic Integration Rules:

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C]=0$	$\int 0dx = C$
$\frac{d}{dx}[kx]=k$	$\int kdx = kx + C$
$\frac{d}{dx}[kf(x)]=kf'(x)$	$\int kf(x)dx = k\int f(x)dx$
$\frac{d}{dx}[f(x)\pm g(x)]=f'(x)\pm g'(x)$	$\int [f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
$\frac{d}{dx}[x^n]=nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1}+C \quad n\neq -1$
$\frac{d}{dx}\sin x=\cos x$	$\int \cos xdx = \sin x + C$
$\frac{d}{dx}\cos x=-\sin x$	$\int \sin xdx = -\cos x + C$
$\frac{d}{dx}\tan x=\sec^2 x$	$\int \sec^2 xdx = \tan x + C$
$\frac{d}{dx}\sec x=\sec x\tan x$	$\int \sec x\tan xdx = \sec x + C$
$\frac{d}{dx}\cot x=-\csc^2 x$	$\int \csc^2 xdx = -\cot x + C$
$\frac{d}{dx}\csc x=-\csc x\cot x$	$\int \csc x\cot xdx = -\csc x + C$

Ex: a. $\int 3xdx$ **b.** $\int \frac{1}{x^3}$ **c.** $\int \sqrt{x}dx$ **d.** $\int 2\sin xdx$ **e.** $\int dx$

f. $\int (x+2)dx$ **g.** $\int 3x^4 - 5x^2 + x)dx$ **h.** $\int \frac{x+1}{\sqrt{x}}dx$ **i.** $\int \frac{\sin x}{\cos^2 x}dx$

Area:

Objective: Use sigma notation to write and evaluate a sum. Understand the concept of area. Approximate the area of a plane region. Find the area of a plane region using limits.

Sigma Notation:

The sum of n terms $a_1, a_2, a_3, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

where i is the index of summation, a_i is the ith term of the sum, and the upper and lower bounds of summation are n and 1.

Ex: a. $\sum_{i=1}^{6} i$ **b.** $\sum_{k=1}^{n} \frac{1}{n}(k^2+1)$ **c.** $\sum_{i=1}^{n} f(x_i)\Delta x$

Properties of Summations:

1. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

2. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Summation Formulas:

1. $\sum_{i=1}^{n} c = cn$

2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Ex: Evaluate $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for n = 10, 100, 1000, 10000

Area of a Plane Region

Use five rectangles to find *two* approximations of the area of the region lying between the graph of $f(x) = x^2$ and the **x-axis** between **x = 0** and **x = 2**.

Rectangles outside the curve are called **circumscribed rectangles** and the sum of the areas is called the **upper sum.**

Rectangles inside the curve are called **inscribed rectangles** and the sum of the areas is called the **lower sum.**

For any region under a curve f bounded by the **x–axis** between **x = a** and **x = b**.

(1) The left end of the rectangle touches the curve = $\sum_{i=1}^{n} f(m_i)\Delta x$

(2) The right end of the rectangle touches the curve = $\sum_{i=1}^{n} f(M_i)\Delta x$

where

- $\Delta x = \dfrac{b-a}{n}$,n is the number of subintervals
- ***f*** **(m_i)** $= f(a + (i - 1)\Delta x)$
- ***f*****(M_i)** $= f(a + (i)\Delta x)$

if the function in increasing or decreasing will change whether (1) or (2) are upper or lower sums

f (m$_i$) is an upper sum if f is decreasing and a lower if f is increasing
f(M$_i$) is a lower sum if f is decreasing and an upper if f is increasing

Limits of the Lower and Upper Sums:
Let f be continuous and nonnegative on the interval [a,b]. The limits as n→∞ of both the lower and upper sums exists and are equal to each other.

Definition of the Area of a Region in the Plane:
Let f be continuous and **nonnegative** on the interval [a,b]. The area of the region bounded by the graph of f, the x-axis, and the vertical lines x = a and x = b is

$$Area = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} < c_i < x_i$$

let $c_i = a + i\Delta x$

Ex: Find the area of the region bounded by the graph $f(x) = x^3$, the **x–axis**, and the vertical lines **x = 0** and **x = 1**.

Riemann Sums and Definite Integrals

Objective: Understand the definition of a Riemann sum. Evaluate a definite integral using limits. Evaluate a definite integral using properties of definite integrals.

Definition of Riemann Sum:

Let f be defined on the closed interval [a,b], and let Δ be a partition of [a,b] given by

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

where Δx_i is the width of the *i* th subinterval. If c_i is any point in the *i* th subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i, \qquad x_{i-1} < c_i < x_i$$

is called the **Riemann Sum** of f for the partition Δ

Definition of a Definite Integral:

If f is defined on the closed interval [a,b] and the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

exists, then *f* is integrable on [a,b] and the limit is denoted by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$$

The limit is called the definite integral of *f* from *a* to *b*. The number a is the **lower limit** of integration and the number b in the **upper limit** of integration.

Notice the similarities between the definite integral and the indefinite integral. Even though they are similar there is a major difference the definite integral results in a number and the indefinite integral results in a family of functions.

Ex: Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$ remember $x_i = \Delta x = \frac{b-a}{n}$ and

$c_i = a + i(\Delta x)$

Continuity Implies Integrability:

If a function f is continuous on the closed interval [a,b], then f is integrable on [a,b].

The Definite Integral as the Area of a Region:
If f is continuous and **nonnegative** on the closed interval [a,b], then the area of the region bounded by the graph of f, the x-axis, and the vertical lines x = a and x = b is given by

$$Area = \int_a^b f(x)dx$$

Ex: Sketch the region corresponding to the definite integral: $\int_1^3 4dx$

Definitions of Two Special Integrals:
1. If *f* if defined at x = a, then we define $\int_a^a f(x)dx = 0$
2. If *f* is integrable on [a,b], then we define
$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Additive Interval Property:
If f is integrable on the three closed intervals [a,c],[c,b], and [a,b] then,
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Properties of Definite Integrals:
If f and g are integrable on [a,b] and *k* is constant, then the functions of *kf* and *f* ± *g* are integrable on [a,b], and
1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

The Fundamental Theorem of Calculus

Objective: Evaluate a definite integral using the Fundamental Theorem of Calculus. Understand and use the Mean Value Theorem for Integrals. Find the average value of a function over a closed interval. Understand and use the Second Fundamental Theorem of Calculus.

We have looked at two major branches of calculus: differential calc (tangent line problem) and integral calc. (area problem). Even though the two seem unrelated there is a connection called the **Fundamental Theorem of Calculus**.

The Fundamental Theorem of Calculus
If a function f is continuous on the closed interval [a,b] and F is an antiderivative of f on the interval [a,b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Using the Fundamental Theorem of Calculus

1. Provided you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of sum.

2. When applying the fundamental Theorem of Calculus, the following notation is convenient.

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\begin{aligned}\int_a^b f(x)dx &= \left[F(x)+C\right]_a^b \\ &= \left[F(b)+C\right] - \left[F(a)+C\right] \\ &= F(b) - F(a)\end{aligned}$$

Ex: Evaluate each definite integral

a. $\int_1^2 (x^2 - 3)dx$ **b.** $\int_1^4 \sqrt{x}dx$ **c.** $\int_0^{\pi/4} \sec^2 x dx$ **d.** $\int_0^2 |2x-1|dx$

Ex: Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines x = 0 and x = 2.

The Second Fundamental Theorem of Calculus:

If *f* is continuous on an open interval I containing *a*, then, for every *x* in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Ex: Evaluate $\frac{d}{dx}\int_0^x \cos t dt$

Integration by Substitution

Objective: Use pattern recognition to find an indefinite integral. Use a change of variables to find an indefinite integral. Use the General Power Rule for Integration to find an indefinite integral. Use a change of variables to evaluate a definite integral. Evaluate a definite integral involving an even or odd function

Pattern Recognition:

We will look at integrating composition functions in two ways *pattern recognition* and *change of variables*.

Remember the Chain Rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$

Anti-differentiation of a Composite Function:

Let g be a function whose range is an interval I, and let f be a function that is continuous on I, If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

If $u = g(x)$ then $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C$$

Recognize the patterns that the following fit $f(g(x))g'(x)$

Ex: **a.** $\int 2x(x^2+1)^4 dx$ **b.** $\int 3x^2\sqrt{x^3+1}dx$ **c.** $\int \sec^2 x(\tan x + 3)dx$

d. $\int (x^2+1)^2 2xdx$ **e.** $\int 5\cos 5xdx$ **f.** $\int x(x^2+1)^2 dx$

Making a Change of Variables

1. Choose a substitution u = g(x). Usually, it is best to choose the inner part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)dx$
3. Rewrite the integral on terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by g(x) to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

Ex: **a.** $\int \sqrt{2x-1}dx$ **b.** $\int x\sqrt{2x-1}dx$ **c.** $\int \sin^2 3x\cos 3xdx$

General Power Rule for Integration:

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

Equivalently, if u = g(x), then

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

Ex: **a.** $\int 3(3x-1)^4 dx$ **b.** $\int (2x+1)(x^2+x)dx$ **c.** $\int 3x^2\sqrt{x^3-2}dx$

d. $\int \frac{-4x}{(1-2x^2)^2}dx$ **e.** $\int \cos^2 x\sin xdx$

<u>Change of Variables for Definite Integrals</u>

If the function u = g(x) has a continuous derivative on the closed interval [a,b] and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Ex: **a.** $\int_0^1 x(x^2+1)^3 dx$ **b.** $\int_1^5 \frac{x}{\sqrt{2x-1}}dx$